1 Linear Algebra Problems Solutions

1. Let A^* be the conjugate transpose of the complex matrix A, i.e., $A^* = \left(\bar{A}\right)^t$. A is said to be Hermitian if $A^* = A$, real symmetric if A is real and $A^t = A$, skew-Hermitian if $A^* = -A$ and normal if $A^*A = AA^*$.

Find the dimension and a basis for each of the following vector spaces.

(a) $M_n(\mathbb{C})$, $n \times n$ complex matrices, over $\mathbb{C}$.

(b) $M_n(\mathbb{C})$ over $\mathbb{R}$

(c) $H_n(\mathbb{C})$, $n \times n$ Hermitian matrices, over $\mathbb{R}$

(d) $H_n(\mathbb{R})$, $n \times n$ real symmetric matrices, over $\mathbb{R}$

(e) $S_n(\mathbb{C})$, $n \times n$ skew-Hermitian matrices, over $\mathbb{R}$

(f) The space consisting of all real polynomials of A over $\mathbb{R}$, where

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{bmatrix}, \omega = \frac{-1+\sqrt{3}i}{2}$$

Is $H_n(\mathbb{C})$ a vector space over $\mathbb{C}$? IS the set of $n \times n$ normal matrices a subspace of $M_n(\mathbb{C})$? Show that $M_n(\mathbb{C}) = H_n(\mathbb{C}) + S_n(\mathbb{C})$, i.e., any $n \times n$ matrix is a sum of Hermitian matrix and a skew-Hermitian matrix.

ANSWER: Note E_{ij} is the $n \times n$ matrix with 1 in the ij position and 0 elsewhere.

(a) $E_{st,}$ $1 \leq s, t \leq n$ for a basis. $\dim = n^2$.

(b) E_{st}, iE_{st}, $1 \leq s, t \leq n$ form a basis. Dimension is $2n^2$.

(c) $E_{st} + E_{ts}$, $s \leq t$ and $i(E_{st} - E_{ts})$, $s < t$, for a basis with dimension n^2.

(d) $E_{st} + E_{ts}$, $s \leq t$, form a basis. Dimension is $\dfrac{n(n+1)}{2}$

(e) $E_{st} - E_{ts}$, $s < t$ and $i(E_{st} + E_{st})$, $s \leq t$, form a basis. Dimension is n^2.

(f) $\left\{I, A, A^2\right\}$ is a basis. Dimension is 3.

$H_n(\mathbb{C})$ and the set of normal matrices are not vector spaces. To see that $M_n(\mathbb{C}) = H_n(C) + S_n(\mathbb{C})$, write

$$A = \frac{A + A^*}{2} + \frac{A - A^*}{2}$$

2. Find the space of matrices commuting with

(a) $A = I_n$

(b) $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

(c) $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, $a \neq b$

(d) $A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

(e) All $n \times n$ matrices

ANSWER: (a) All $n \times n$ matrices

(b) All the matrices of the form $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$

(c) All the matrices of form $\begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}$

(d) All the matrices of form $\begin{pmatrix} a & b & c & d \\ 0 & a & b & c \\ 0 & 0 & a & b \\ 0 & 0 & 0 & a \end{pmatrix}$

(e) All the $n \times n$ scalar matrices cI_n.

3. True or False. If true, what is the dimension? Basis?
(a) $\left\{(x, y) : x^2 + y^2 = 0, x, y \in \mathbb{R}\right\}$ is a subspace of $\mathbb{R}^2$.
(b) $\left\{(x, y) : x^2 + y^2 = 0, x, y \in \mathbb{C}\right\}$ is a subspace of $\mathbb{C}^2$.
(c) $\left\{(x, y) : x^2 - y^2 = 0, x, y \in \mathbb{R}\right\}$ is a subspace of $\mathbb{R}^2$.
(d) $\{(x, y) : x - y = 0, x, y \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^2$.
(e) $\{(x, y) : x - y = 1, x, y \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^2$.
(f) $\{p(x) : p(x) \in \mathbb{P}[x] \text{ has degree } 3\}$ is a subspace of $\mathbb{P}[x]$.
(g) $\{p(x) : p(0) = 0, p(x) \in \mathbb{P}[x]\}$ is a subspace of $\mathbb{P}[x]$
(h) $\{p(x) : 2p(0) = p(1)\}$ is a subspace of $\mathbb{P}[x]$.
(i) $\{p(x) : p(x) \geq 0, p(x) \in \mathbb{P}[x]\}$ is a subspace of $\mathbb{P}[x]$.
ANSWER: (a) True (b) False (c) False (d) True (e) False (f) False (g) True (h) True (i) False.

4. Show that $M_2(\mathbb{R}) = W_1 \oplus W_2$, where

$$W_1 = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a, b \in \mathbb{R} \right\}$$

$$W_2 = \left\{ \begin{pmatrix} c & d \\ d & -c \end{pmatrix} : c, d \in \mathbb{R} \right\}.$$

ANSWER: One can see that

$$\begin{pmatrix} x & y \\ u & v \end{pmatrix} = \begin{pmatrix} \frac{x+v}{2} & \frac{y-u}{2} \\ -\frac{y-u}{2} & \frac{x+v}{2} \end{pmatrix} + \begin{pmatrix} \frac{x-v}{2} & \frac{y+u}{2} \\ \frac{y+u}{2} & -\frac{x-v}{2} \end{pmatrix} \text{ and}$$

$W_1 \cap W_2 = \{0\}$

5. Let A be an $n \times n$ real matrix.
(a) Show that if $A^t = -A$ and n is odd, then $|A| = 0$.
(b) Show that if $A^2 + I = 0$, then n must be even.
(c) Does (b) remain true for complex matrices?
ANSWER: (a) $|A| = |A^t| = |-A| = (-1)^n |A| = -|A|$ if n is odd.
(b) $|A|^2 = \left|A^2\right| = |-I| = (-1)^n$. If n is odd, then $|A|^2 = -1$. This is impossible when A is a real matrix.
(c) No.

6. Introduce the correspondence between complex numbers and real matrices:

$$z = x + iy \tilde{\ } Z = \begin{pmatrix} x & y \\ -y & x \end{pmatrix} \in M_2(\mathbb{R}),$$

and define for each pair of complex numbers u and v :

$$q = (u, v) \cong C(q) = \begin{pmatrix} u & v \\ -\bar{v} & \bar{u} \end{pmatrix} \in M_2(\mathbb{C})$$

(a) Show that $\bar{z}\tilde{\ }Z^t$.

(b) Show that $ZW = WZ$

(c) Show that $z\tilde{\ }Z$ and $w\tilde{\ }W$ imply $zw + ZW$.

(d) Find Z^n, where $z = r(\cos\theta + i\sin\theta)$.

(e) What is the matrix corresponding to i?

(f) Show that $|C(q)| \geq 0$. Find $C(q)^{-1}$ when $|u|^2 + |v|^2 = 1$.

(g) Replace each entry of $C(q)$ with the corresponding 2×2 real matrix to the entry to get

$$\mathcal{R}(q) = \begin{pmatrix} U & V \\ -V^t & U^t \end{pmatrix} \in M_4(.)$$

Then $|\mathcal{R}(q)| \geq 0$.

(h) Show that $\mathcal{R}(q)$ is similar to a matrix of form

$$\begin{pmatrix} U & X \\ -X & U \end{pmatrix}$$

for some X.

(i) Show that $\mathcal{R}(q)$ is singular if and only if $\mathcal{C}(q)$ is singular if and only if $u = v = 0$.

ANSWER: It is routine to verify (a), (b), and (c).

(d) Since $z^n = r^n(\cos n\theta + \sin n\theta)$,

$$Z^n = r^n \begin{pmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{pmatrix}$$

(e) $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

(f) $|\mathcal{C}(q)| = |u|^2 + |v|^2 \geq 0$ and

$$\mathcal{C}(q)^{-1} = \begin{pmatrix} \bar{u} & -v \\ \bar{v} & u \end{pmatrix}, \text{ when } |\mathcal{C}(q)| = 1.$$

(g) Write $u = u_1 + iu_2$ and $v = v_1 + iv_2$ to get $\mathcal{R}(q)$. Note that U and $-V^t$ commute. By computation

$$|\mathcal{R}(q)| = |UU^t + V^tV| = \left(|u^2| + |v|^2\right)^2 \geq 0$$

(h) Exchange the last two rows and columns of $\mathcal{R}(q)$.

(i) From (g), $|\mathcal{R}(q)| = 0 \iff |\mathcal{C}(q)| = 0 \iff u = v = 0$

7. True or false

(a) For any $m \times n$ matrix A with rank r, there exists invertible $m \times m$ and $n \times n$ matrices P and Q such that

$$A = P \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} Q.$$

(b) For any $n \times n$ matrix A with rank r, there exists an invertible $n \times n$ matrix P such that

$$A = P \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} P^{-1}.$$

(c) If A is a real matrix and A^{-1} exists over $\mathbb{C}$, then A^{-1} is also a real matrix.

(d) If $(A^*)^2 = A^2$, then $A^* = A$ or $A^* = -A$.

(e) If $\operatorname{rank} A = \operatorname{rank} B$, then $\operatorname{rank}\left(A^2\right) = \operatorname{rank}\left(B^2\right)$
(f) $\operatorname{rank}(A+B) \leq \operatorname{rank}(A) + \operatorname{rank}(B)$
(g) $\operatorname{rank}(A-B) \leq \operatorname{rank}(A) - \operatorname{rank}(B)$
(h) Since $(1,i)$ and $(i,-1)$ are linearly independent over $\mathbb{R}$, the matrix $\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}$ is invertible.

ANSWER: (a)True (b)False (c) True (d)False (e)False (f)True (g)False (h)False

8. Suppose that A and B are both $m \times n$ matrices. Show that $Ax = 0$ and $Bx = 0$ have the same solution space if and only if there exists an invertible matrix C such that $A = CB$. Use this fact to show that if $\operatorname{rank}\left(A^2\right) = \operatorname{rank}(A)$, then there exists an invertible matrix D such that $A^2 = DA$.

ANSWER: it is sufficient to show that there is an invertible matrix C such that $A = CB$ when $Ax = 0$ and $Bx = 0$ have the same solution space.

First notice that A and B must have the same rank. Let P_1 and P_2 be the permutation matrices such that

$$P_1 A = \begin{pmatrix} A_1 \\ Q_1 A_1 \end{pmatrix} \text{ and } P_2 B = \begin{pmatrix} B_1 \\ Q_2 B_1 \end{pmatrix},$$

where A_1 and B_1 are respectively $r \times r$ submatrices of A and B with rank r, and Q_1 and Q_2 are some matrices os size $(m-r) \times n$. $Ax = 0$ and $Bx = 0$ have the same solution space if and only if $A_1 x = 0$ and $B_1 x = 0$ have the same solution space. We observe that

$$\operatorname{rank}\left[\begin{pmatrix} A_1 \\ B_1 \end{pmatrix}\right] = \operatorname{rank}(B_1),$$

and thus know that there is an $r \times r$ invertible matrix C_1 such that $A_1 = C_1 B_1$. Thus

$$P_1 A = \begin{pmatrix} A_1 \\ Q_1 A_1 \end{pmatrix} = \begin{pmatrix} C_1 & 0 \\ Q_1 C_1 & I_{m-r} \end{pmatrix} \begin{pmatrix} B_1 \\ Q_2 B_1 \end{pmatrix} = C_2 P_2 B$$

where

$$C_2 = \begin{pmatrix} C_1 & 0 \\ Q_1 C_1 - Q_2 & I_{m-r} \end{pmatrix}$$

is of full rank. Take $C = P_1^{-1} C_2 P_2$.

If $\operatorname{rank}\left(A^2\right) = \operatorname{rank}(A)$, then $A^2 x = 0$ and $Ax = 0$ have the same solution space. It follows from the above result that $A^2 = DA$ for some invertible matrix D.

9. What are the matrices that are similar to themselves only?

ANSWER: Scalar matrices cI. Consider $P^{-1}AP = A$ or $PA = AP$.

10. Prove assertions (a) and (b) and construct an example for (c).

(a) Let $A \in M_n(\mathbb{C})$. If the eigenvalues of A are distinct from each other, then A is diagonalizable, i.e., there is an invertible matrix P such that $P^{-1}AP$ is diagonal.

(b) If matrix A commutes with a matrix with distinct eigenvalues, then A is diagonalizable.

(c) Give an example of a matrix A that is diagonalizable but not unitary diagonalizable, that is, $P^{-1}AP$ is diagonal for some invertible P, but U^*AU is not diagonal for any unitary matrix U.

ANSWER: (a) Let $u_1, u_2, ..., u_n$ be the eigenvectors of A belonging to the eigenvalues $\lambda_1, ..., \lambda_n$, respectively, and $\lambda_i \neq \lambda_j$ when $i \neq j$. We first show by induction that $u_1, u_2, ..., u_n$ are linealry independent.

Let

$$a_1 u_1 + ... + a_n u_n = 0$$

and apply A to the above equation to get

$$a_1 \lambda_1 u_1 + a_2 \lambda_2 u_2 + ... + a_n \lambda_n u_n = 0$$

On the other hand,

$$a_1 \lambda_n u_1 + ... + a_n \lambda_n u_n = 0.$$

Subtracting,

$$a_1 (\lambda_1 - \lambda_n) u_1 + ... + a_{n-1} (\lambda_{n-1} - \lambda_n) u_{n-1} = 0.$$

By induction, $u_1, u_2, ..., u_{n-1}$ are linearly independent and

$$a_1 = a_2 = ... = a_{n-1} = 0$$

since $\lambda_i \neq \lambda_j$ for $i \neq j$, consequently $a_n = 0$.

Now set $P = (u_1, u_2, ..., u_n)$. Then P is an invertible matrix and

$$AP = P diag \{\lambda_1, ..., \lambda_n\},$$

that is, $P^{-1}AP$ is a diagonal matrix.

(b) It suffices to show that if A commutes with a diagonal matrix whose diagonal entries are distinct, then A must be diagonal. This can be seen by a direct computation.

(c) Take $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$

11. True or false

(a) If $A^k = 0$ for all positive integers $k \geq 2$, then $A = 0$.

(b) If $A^k = 0$ for some integer k, then $\operatorname{tr} A = 0$.

(c) If $\operatorname{tr} A = 0$, then $|A| = 0$.

(d) If A and B are similar, then $|A| = |B|$

(e) If A and B are similar, then they have the same eigenvalues.

(f) If A and B have the same eigenvalues, then they are similar.

(g) If A and B have the same characteristic polynomial, then they have the same eigenvalues.

(h) If A and B have the same eigenvalues, then they have the same characteristic polynomial.

(i) If A and B have the same characteristic polynomial, then they are similar.

(j) If $\operatorname{tr} A^k = \operatorname{tr} B^k$ for all positive integers k, then $A = B$.

(k) If the eigenvalues of A are $\lambda_1, ..., \lambda_n$, then A is similar to the diagonal matrix $diag \{\lambda_1, ..., \lambda_n\}$.

(l) $diag \{1, 2, ..., n\}$ is similar to $diag \{n, ..., 2, 1\}$

(m) If A has a repeated eigenvalue, then A is not diagonalizable.

(n) If A is diagonalizable, then A is normal.

(o) If A is unitarily diagonalizable, then A is normal.

(p) If A has r nonzero eigenvalues, then $\operatorname{rank}(A) \geq r$.

ANSWER: (a)False (b)True (c)False (d)True (e)True (f)False (g)True (h)True (i)False

12. Let $A \in M_n(\mathbb{C})$ and $A \neq 0$. Define a transformation on $M_n(\mathbb{C})$ by

$$\mathcal{T}(X) = AX - XA,\ X \in M_n(\mathbb{C})$$

Show that

(a) $\mathcal{T}$ is linear

(b) Zero is an eigenvalue of T

(c) If $A^k = 0$, then $\mathcal{T}^{2k} = 0$.

(d) If A is diagonalizable, so is $\mathcal{T}$.

(e) $\mathcal{T}(XY) = X\mathcal{T}(Y) + \mathcal{T}(X)Y$.

(f) If A and B commute, so do $\mathcal{T}$ and $\mathcal{L}$ where $\mathcal{L}$ is defined as

$$\mathcal{L}(X) = BX - XB,\ X \in M_n(\mathbb{C}).$$

Find all A such that $\mathcal{T}$ =0 and discuss the converse of (f).

ANSWER: (a) By direct verification

(b) $\mathcal{T}(A) = 0$

(c) Compute $\mathcal{T}^2(X), \mathcal{T}^3(X), ...$ it can be seen that each term of $\mathcal{T}^{2k}(X)$ contains a factor of A^m, $m \leq k$. Thus $\mathcal{T}^{2k} = 0$.

(d) Let

$$P^{-1}AP = diag\{\lambda_1, ..., \lambda_n\}$$

and let P_i be the i^{th} column of P. Then

$$AP_i = \lambda_i P_i,\ i = 1, 2, ..., n.$$

Let B_{ij} be the matrix having P_i as its j^{th} column and 0 as other columns. Then $\{B_{ij}\}$ form a basis for $M_n(\mathbb{C})$ and $\mathcal{T}$ has the matrix representation on the basis

$$T = \begin{pmatrix} \lambda_1 I - A^t & 0 & \cdots & 0 \\ 0 & \lambda_2 I - A^t & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n I - A^t \end{pmatrix}$$

It is easily noted that if A is diagonalizable, so is $\mathcal{T}$.

(e) By a direct verification.

(f) If $\mathcal{T}$ and $\mathcal{L}$ commute, then $\mathcal{TL}(X) = \mathrm{LT}(X)$ is equivalent to

$$ABX + XBA = BAX + XAB$$

or

$$(AB - BA)X = X(AB - BA)$$

When A and B commute, $AB - BA = 0$. So $\mathcal{T} = 0$ if and only if A is a scalar matrix, cI_n.

If $\mathcal{T}$ commutes with $\mathcal{L}$, then $AB - BA$ commutes with any matrix in $M_n(\mathbb{C})$. Thus $AB - BA$ is a scalar matrix. On the other hand $\mathrm{tr}(AB - BA) = 0$, so $AB = BA$.

13. Let W be an invariant subspace of a linear transformation $\mathcal{A}$ on a finite-dimensional vector space V.

(a) Show that if $\mathcal{A}$ is invertible, then W is also invariant under $\mathcal{A}^{-1}$.

(b) If $V = W \oplus W'$, is W' necessarily invariant under A?

ANSWER: (a) Apply $\mathcal{A}^{-1}$ to both sides of $\mathcal{A}(W) \subseteq W$ to get

$$W \subseteq \mathcal{A}^{-1}(W).$$

On the other hand,

$$\dim \mathcal{A}^{-1}(W) \leq \dim W,$$

Therefore

$$W = \mathcal{A}^{-1}(W).$$

(b) No, in general.

14. Show that if A is an invertible Hermitian matrix, then there exists and invertible matrix P such that $P^*AP = A^{-1}$.

ANSWER: Consider the case when A is real diagonal.

15. Is it possible for some non-Hermitian matrix $A \in M_n(\mathbb{C})$ to satisfy $x^*Ax \geq 0$ for all $x \in \mathbb{R}^n$? $x \in \mathbb{C}^n$?

ANSWER: Yes, when $x \in \mathbb{R}^n$. For instance, $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. No when $x \in \mathbb{C}^n$. In this case A must be Hermitian.

16. Construct examples

(a) Matrices A and B that have only positive eigenvalues, AB has only negative eigenvalues. (Note that A and B are not necessarily Hermitian).

(b) Is it possible that $A + B$ has only negative eigenvalues for matrices A and B with positive eigenvalues?

(c) Matrices A, B, and C are positive definite, ABC has only negative entries.

(d) Is it possible that the matrices in (c) are 3×3?

ANSWER: (a) Take $A = \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -5 \\ 0 & 1 \end{pmatrix}$

(b) No, because $\operatorname{tr}(A + B) = \operatorname{tr} A + \operatorname{tr} B > 0$.

(c) Take

$$A = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}, \ B = \begin{pmatrix} 3 & -1 \\ -1 & 1 \end{pmatrix}, \text{ and } C = \begin{pmatrix} 1 & 0 \\ 0 & 30 \end{pmatrix}.$$

Then A, B, and C are positive definite, the eigenvalues of ABC are $-5, -12$.

(d) No. Note that $|ABC| > 0$.

17. Let $A \in M_n(\mathbb{C})$ be a normal matrix. Show that

(a) $\ker A^* = \ker A$

(b) $\operatorname{Im} A^* = \operatorname{Im} A$

(c) $\mathbb{C}^n = \operatorname{Im} A \oplus \ker A$

ANSWER: (a) It is sufficient to show that $\ker A^* \subseteq \ker A$. The other way around is similar. Let

$$x \in \ker A^* \text{ or } A^*x = 0,$$

then $AA^*x = 0$ and $A^*Ax = 0$ as A is normal. Thus $x^*A^*Ax = (Ax)^*(Ax) = 0$ and $ax = 0$.

(b) Let $x \in \operatorname{Im} A^*$ and $x = A^*y$. Since A is normal, assume $A^* = AU$ for some unitary matrix U, then

$$x = A^*y = AUy \in \operatorname{Im} A.$$

Thus $\operatorname{Im} A^* \subseteq \operatorname{Im} A$. The other direction is similar.

(c) It suffices to show that $\operatorname{Im} A^* \cap \ker A = \{0\}$. Let $x = A^*y$ and $Ax = 0$. Then

$$0 = y^*Ax = y^*AA^*y$$

and $A^*y = 0$, i.e., $x = 0$.

18. A permutation matrix is a matrix which has exactly one 1 in each row and each column.

(a) How many $n \times n$ permutation matrices are there?

(b) The product of two permutation matrices of the same size is also a permutation matrix. How about the sum?

(c) Show that any permutation matrix is invertible and its inverse is equal to its transpose.

(d) For what permutation matrices P, does $P^2 = I$?

ANSWER: (a) $n!$, the size of S_n, the symmetric group on n objects.

(b) By a direct verification. No.

(c) If P is a permutation matrix, then $PP^t = I$.

(d) Symmetric permutation matrices.

19. Let P be the $n \times n$ permutation matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} = \begin{pmatrix} 0 & I_{n-1} \\ 1 & 0 \end{pmatrix}$$

Then show the following

(a) For any positive integer $k \le n$,

$$P^k = \begin{pmatrix} 0 & I_{n-k} \\ I_k & 0 \end{pmatrix}, \text{ and}$$

$$P^{n-1} = P^t,\ P^n = I_n$$

(b) $P, P^2, ..., P^n$ are linearly independent.

(c) $P^i + P^j$ is a normal matrix

(d) For $n \ge 3$, P is diagonalizable over $\mathbb{C}$, but not over $\mathbb{R}$.

(e) For every P^i, there exists a permutation matrix T such that $T^{-1}P^iT = P$.

ANSWER: (a) By direct computations. Note that for any matrix A, AP is the matrix obtained from A by moving the last column of A to the first column.

(b) Consider

$$k_1P + k_2P^2 + ... + k_nP^n = 0$$

Since the $k's$ are in different positions of the matrix on the left hand side, it follows that all the $k's$ must be equal to zero.

(c) It is routine to check.

(d) $i + j$ is divisible by n.

(e) Note the the characteristic polynomial of P is $\lambda^n - 1$.

(f) Take $T = (t_{ij})$ to be the permutation matrix with $t_{k,k'} = 1$, $k = 1, 2, ..., n$, and 0 otherwise, where k' is the positive integer such that

$$0 < k' = (k-1)\,i + 1 - mn \leq n$$

for some nonnegative integer m. For instance, if $n = 5$ and $i = 3$, then

$$t_{11} = t_{24} = t_{32} = t_{45} = t_{53} = 1$$

and 0 otherwise. It is easy to check that $T^{-1}P^3T = P$. This is a well-known result from group theory.

20. Let $\mathcal{A}$ be a linear transformation on an inner product space V. Show that for any unit vector $x \in V$

$$(Ax, x)\,(x, Ax) \leq (Ax, Ax)\,.$$

In particular, for $A \in M_n(\mathbb{C})$ and $x \in \mathbb{C}^n$ with $||x|| = 1$,

$$x^*A^*xx^*Ax \leq x^*A^*Ax.$$

ANSWER: Recall the Cauchy-Scharz inequality

$$|(x, y)|^2 \leq (x, x)\,(y, y)$$

or

$$(x, y)\,(y, x) \leq (x, x)\,(y, y)$$

So if x is a unit vector,

$$(x, y)\,(y, x) \leq (y, y)\,.$$

Replace y by $\mathcal{A}x$,

$$(\mathcal{A}x, x)\,(x, \mathcal{A}x) \leq (\mathcal{A}x, \mathcal{A}x)\,.$$

When $A \in M_n(\mathbb{C})$ and $x \in \mathbb{C}^n$ with $x^*x = 1$,

$$x^*Axx^*A^*x \leq x^*A^*Ax.$$

21. Let $e_1, e_2, ..., e_n$ be vectors of an inner product space over a field $\mathbb{F}$, and let $A = (a_{ij})$, where

$$a_{ij} = (e_i, e_j)\,,\ i, j = 1, 2, ..., n.$$

Show that $e_1, e_2, ..., e_n$ are linearly independent if and only if A is nonsingular.

ANSWER: Let

$$\lambda_1 e_1 + ... + \lambda_n e_n = 0.$$

Take the inner product of both sides with e_i, $i = 1, 2, ..., n$. Then consider the linear equation system of $\lambda_1, \lambda_2, ..., \lambda_n$.

22. Let V be an inner product space over $\mathbb{R}$.

(a) If e_1, e_2, e_3 are three vectors in V with pairwise product negative, that is, $(e_i, e_j) < 0$, $i, j = 1, 2, 3$, $i \neq j$,show that e_1, e_2, e_3 are linearly independent.

(b) Is it possible for three vectors in the xy-plane to have pairwise negative products?

(c) Does (a) remain valid when the word "negative" is replaced with "positive"?

(d) Suppose that u, v, and w are three unit vectors in the xy-plane. What are the maximum and minimum values that

$$(u, v) + (v, w) + (w, u)$$

can attain? and when?

ANSWER: (a) Suppose that e_1, e_2, e_3 are linealry dependent. It may be assume that e_1, e_2, and e_3 are unit vectors and that

$$e_3 = \lambda_1 e_1 + \lambda_2 e_2.$$

Then

$$(e_1, e_3) = \lambda_1 + \lambda_2 (e_1, e_2) < 0$$

and

$$(e_2, e_3) = \lambda_1 (e_1, e_2) + \lambda_2 < 0.$$

Thus when $(e_1, e_2) < 0$

$$\lambda_2 < \lambda_1 [-(e_1, e_2)] < \lambda_2 [-(e_1, e_2)]^2.$$

Hence $(e_1, e_2)^2 > 1$. This is impossible when e_1 and e_2 are unit vectors.

(b) No, since the dimension of the xy-plane is 2.

(c) No.

(d) $-\frac{3}{2}, 3$. The maximum is attained when $u = v = w$ and the minimum is attained when the angles between any two of them are equal to $\frac{2\pi}{3}$.

23. If $\{e_1, ..., e_n\}$ is an orthonormal basis for an inner product space V over $\mathbb{C}$, and $x \in V$, show that

$$x = \sum_{i=1}^{n} (x, e_i) e_i$$

and

$$(x, x) \geq \sum_{i=1}^{k} |(x, e_i)|^2, \; 1 \leq k \leq n.$$

Why are pairwise orthogonal nonzero vectors linearly independent?

ANSWER: Let $x = \lambda_1 e_1 + ... + \lambda_n e_n$. Taking the inner product of both sides with e_i we get:

$$\lambda_i = (x, e_i), \; i = 1, 2, ..., n.$$

Then by a direct computation we have

$$(x, x) = |\lambda_1|^2 + ... + |\lambda_n|^2$$

24. Let W be a subspace of an inner product space V and let S be a subset of V. Answer true or false.

(a) There is a unique subspace W' such that $W' + W = V$.

(b) There is a unique subspace W' such that $W' \oplus W = V$.

(c) There is a unique subspace W' such that $W' \oplus W = V$ and $(w, w') = 0$ for all $w \in W$ and $w' \in W'$/

(d) $(W^{\perp})^{\perp} = W$.

(e) $\left(S^{\perp}\right)^{\perp} = S$

(f) $\left[\left(S^{\perp}\right)^{\perp}\right]^{\perp} = S^{\perp}$

(g) $\left(S^{\perp} + W\right)^{\perp} = \left(S^{\perp}\right)^{\perp} \cap W^{\perp}$

(h) $\left(S^{\perp} \cap W\right)^{\perp} = \left(S^{\perp}\right)^{\perp} + W^{\perp}$

ANSWER: (a)False (b)False (c)True (d)True (e)False (f)True (g)True (h)True