More Algebra Questions

Groups

1) What is mean by a group acting on a set? Give two characterizations and show they are equivalent. Give two examples of a group action.
2) Let G be a finite group and let H be a subgroup whose index in G is the smallest prime dividing the order of G. Show that H is a normal subgroup of G.
3) Derive the class equation. Use it to prove that the center of a (nontrivial) p-group is nontrivial.
4) Show that if Z is the center of G and G/Z is cyclic that G is abelian.
5) Show that if p is any prime that a group of order p^2 is abelian.
6) Let G be a group and G' its commutator subgroup, that is G' is the subgroup of G generated by elements of the form $xyx^{-1}y^{-1}$ with $x, y \in G$. Show that
   (a) $G' \lhd G$.
   (b) G/G' is abelian.
   (c) If $H \lhd G$ and G/H is abelian, then $G' \subseteq H$.
7) Show that if G is a group and $N \lhd G$, then G is solvable if and only if N and G/N are solvable.
8) Show that the following two statements are equivalent.
   (a) Every finite group of odd order is solvable.
   (b) Every finite nonabelian simple group has even order.
9) Suppose that p is a prime and that G is a nonabelian group of order p^3. Let Z denote the center of G. Show that:
   (a) $G' = Z$
   (b) G/Z is the direct product of two cyclic groups of order p.
10) Show that a finite abelian group has a subgroup of every order dividing the order of the group.
11) Show that a finite group always has a composition series.
12) SUppose that an arbitrary group G has two subgroups H and K each of finite index in G. Show that the intersection $H \cap K$ has finite index in G; in particular, show that $[G : H \cap K] \leq [G : H]\,[G : K]$.
13) Let G be a finite group, and p a prime dividing the order of G. Let $H = \{x \in G : |x| = p^m, \text{ some } m\}$. Show that H is the union of all the p-Sylow subgroups of G.
14) Show that a finite cyclic group has a unique subgroup of every order dividing the order of the group.
15) State the structure theorem for finitely generated abelian groups. Characterize, up to isomorphism, all abelian groups of order 72 in terms of both elementary divisors and invariant factors.

Rings

1) Define the terms integral domain, irreducible element, and prime element.
   (a) Are there rings with no irreducible elements?
   (b) Are there commutative rings in which irreducible elements are not prime?
   (c) Prove that in a commutative ring, prime elements are irreducible.
2) Give examples of a noncommutative ring with zero divisors, a noncommutative division ring, and integral domain, UFD, PID, Euclidean domain, and examples which show that ID$\not\Rightarrow UFD \not\Rightarrow PID \not\Rightarrow ED$.

3) Determine what is meant by a Noetherian ring. Give three conditions and demonstrate their equivalence.
4) Why isn't every integral domain a UFD? That is what goes wrong when one tries to factor nonzero nonunits? What conditions does one need to impose? Hint: there are separate conditions which guarantee the existence of a factorization and uniqueness of a factorization.
5) What is the characteristic of a commutative ring? What can one say about the characteristic of an integral domain?
6) Prove the ED implies PID.
7) Show that PID implies UFD.
8) Let A be a UFD, and X an indeterminate. Show that any irreducible in A remains irreducible in $A[X]$.
9) Show that $k[x_1, x_2, ...]$ is a non-Noetherian UFD.
10) Prove that over a field K, a polynomial of degree n has at most n roots in any splitting field. Does this remain true if the field K is replaced by a division ring like Hamilton's quaternions? Why or why not.
11) Let A be an integral domain with quotient field K, and let $f \in A[X]$. Show that f is irreducible over A iff f is primitive and irreducible over $K[X]$.
12) Let A be a UFD with quotient field K, and let L be a field. Let $f \in A[X]$, $\deg(f) = r \geq 1$, and let $\sigma : A \to L$ be a ring homomorphism. Show that if $\deg(f^\sigma) = \deg(f)$ and f^σ is irreducible in $L[X]$, then f is irreducible in $K[X]$. In particular, if f is primitive, then f is irreducible in $A[X]$.
13) State the Eisenstein criterion.
14) Show $f(x) = x^3 + 3x^2 + 5x + 2$ is irreducible by using 12) and $\sigma : \mathbb{Z} \to \mathbb{Z}_3$.
15) State the Chinese Remainder Theorem.
16) Let R be a ring, x an indeterminate. For each property listed, consider the question, "If R has property (?) does $R[x]$? If so give a proof, if not a counterexample.

(a) Integral Domain
(b) PID
(c) UFD
(d) Noetherian ring

17) Define the notion of prime and maximal ideals. Give characterizations of each in terms of quotients. Give an example to show that not all prime ideals are maximal.

(a) Is (x) prime / maximal in $\mathbb{Z}[x]$? $\mathbb{Q}[x]$?
(b) If k is a field, is (x) prime / maximal in $k[x, y]$?
(c) Characterize the maximal ideals of $\mathbb{Q}[x]$ and $\mathbb{C}[x]$.
(d) Is $(x - 3)$ prime / maximal in $k[x, y]$?

18) What is meant by localization of a ring with respect to a multiplicative set. What does localizing at a prime do? Describe the localization of the ring $\mathbb{Z}[x]$ at the prime ideal (x). Is (x) maximal in $\mathbb{Z}[x]_{(x)}$? If so, describe the field to which its quotient is isomorphic.

Modules

1) Discuss the notion of a minimal polynomial of a linear transformation. Does it have to be irreducible?
2) Let V be a finite dimensional vector space over a field k and let $T \in End_k(V)$. Show how to use T to make V into a finitely generated torsion $k[x]$-module.
3) State the basic decomposition theorem for finitely generated modules over PIDs and apply it to the situation in the previous question.

4) Explain Rational Canonical Form.
5) Describe Jordan Canonical Form.
6) Find all rational and canonical forms of a matrix in $M_5(\mathbb{C})$ having minimal polynomial $x^2(x-1)$. Be sure to give invariants and characteristic polynomial.
7) Let M be a free module of finite rank over a PID R and N a submodule. What does the elementary divisor theorem say about bases of N versus M?
8) Let M and N be modules over a commutative ring R.

(a) Characterize the tensor product $M \otimes_R N$ in terms of a universal mapping.

(b) Give a construction for the tensor product $M \otimes_R N$.

(c) Show that $\mathbb{Z}_m \otimes_{\mathbb{Z}} \mathbb{Z}_n = 0$ if $\gcd(m,n) = 1$.

(d) Characterize $\mathbb{Z}_m \otimes_{\mathbb{Z}} \mathbb{Z}_n$ in general and give a proof.

(e) If G is a finite abelian group and $G \otimes \mathbb{Z}_p = 0$ for all primes p, show that $G = 0$. Does the result remain true if G is infinite?

Fields and Galois

1) Let E/F be an extension of fields and let $\alpha \in E$. Define what is meant by the phrase α is algebraic over E. What is meant by saying the E/F is an algebraic extension?
2) What is meant by the minimal polynomial of an element algebraic over F? Why is $F[\alpha] \cong F(\alpha)$?
3) If α is algebraic over F, what is the degree of $F(\alpha)/F$? Exhibit a basis and prove it is a basis.
4) Show that an extension field E/F is finite if and only if it is algebraic and finitely generated.
5) Let F be a field, and $f \in F[x]$ be irreducible. Show that all roots of f occur with the same multiplicity.
6) Prove that the Galois group of a finite extension of finite fields is cyclic.
7) Compute the Galois group of $x^8 - 1$ over $\mathbb{Z}_3$. Is $x^8 - 1$ separable over $\mathbb{Z}_3$? Doe you really need to check? Hint: $x^9 - x = x(x^8 - 1)$.
8) Describe the Galois group of $\mathbb{Q}(\zeta_n)/\mathbb{Q}$ and discuss the Galois correspondence. Here ζ_n is a primitive nth root of unity.
9) Describe all the fields which lie between $\mathbb{Q}$ and $\mathbb{Q}(\zeta_{12})$.
10) Show that any finite subgroup of the multiplicative group of a field is cyclic.
11) Descrive the construction of an algebraic closure of a field.
12) Compute the Galois group of $x^6 + 27$ over $\mathbb{Q}$.
13) Compute the Galois group of $x^8 - 16$ over $\mathbb{Q}$.