# 1 Math 511b Test Review

Module Test - Review

1. Show that $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}$ as $\mathbb{Z}$-modules.

2. Show that $\mathbb{Q}$ is not a free $\mathbb{Z}$-module.

3. Describe the abelian group with the presentation

   $$A = \langle a, b, c : 4a + 10b - 8c = 0, 2a + 8b - 4c = 0 \rangle$$

4. Let $F$ be a field and $V$ and $n$ dimensional vector space over $F$. There is an $F$-linear endomorphism $T$ of the tensor product $V \otimes V$ mapping $v \otimes w$ to $T(v \otimes w) = w \otimes v$ for all $v, w \in V$. Determine the eigenvalues of $T$ and further determine bases for corresponding eigenspaces.

5. Proof or counterexample:

   (a) If $R$ is a $PID$ and $M$ is a finitely generatored torsion-free $R$-module then $M$ is free.

   (b) If $R$ is an $ID$ and $M$ is a finitely generated torsion free $R$-module, then $M$ is free.

   (c) Every submodule of a free module is free.

   (d) $R$ is commutative with 1; $M$ an $R$-module implies that $M$ is a finite set if and only if finitely generated and every element is a torsion element.

   (e) If $E$ and $F$ are free $R$-modules, then $E \oplus F$ is free.

6. Find the characteristic polynomial, minimal polynomial, rational cannonical form, and the JCF of $A = \begin{pmatrix} 0 & 4 & 0 \\ 2 & 0 & 8 \\ 0 & -1 & 0 \end{pmatrix}$.

7. Suppose $A$ and $B$ are finitely generated abelian groups. View $A$ and $B$ as $\mathbb{Z}$-modules. Compute $A \otimes_{\mathbb{Z}} B$ as explicitly as possible.

8. Let $0 \to A \to B \to C \to 0$ be an exact sequence of $R$-modules where $R$ is any ring with 1. Prove that if $B$ has torsion elements then either $A$ or $C$ has torsion elements.

9. Let $M$ be a unitary cyclic $R$-module, $R$ a ring with 1. Show that $M \cong R/I$ for some left ideal $I$ in $R$.

10. Let $M$ be an $R$-module and let $A, B, C$ be submodules. If $C \subseteq A$, prove that

    $$A \cap (B + C) = (A \cap B) + C.$$

11. Suppose that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_1 & \xrightarrow{\phi} & M & \xrightarrow{\phi'} & M_2 & \longrightarrow & 0 \\ & & \downarrow_f & & \downarrow_g & & \downarrow_h & & \\ 0 & \longrightarrow & N_1 & \xrightarrow{\psi} & N & \xrightarrow{\psi'} & N_2 & \longrightarrow & 0 \end{array}$$

is a commutative diagram of $R$-modules and $R$-module homomorphisms. Assume that the rows are exact and that $f$ and $h$ are isomorphisms. Prove that $g$ is an isomorphism.

12. Suppose $F$ is a field, $A$ and $B$ are $n \times n$ matrices over $F$ and $A' = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$ is similar to $B' = \begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix}$. Show that $A$ and $B$ are similar over $F$.

13. Use Smith Normal Form to find all integral solutions of the equation

$$\begin{aligned} 2x_1 - 7x_2 + 12x_3 &= 4 \\ -4x_1 + 3x_2 - 2x_3 &= -8 \end{aligned}$$

14. Suppose $T : V \to V$ is a linear transformation on a finite dimensional vector space $V$ over a field $F$, and that $T$ has invariant factors $x-1, x(x-1)$, and $x(x-1)^2$.

    (a) What is $\dim_F V$?

    (b) Is $T$ one-to-one?

    (c) What is the minimal polynomial of $T$.

    (d) What are the $RCF$ and $JCF$?

15. An $n \times n$ matrix $A$ over a field $F$ is called nilpotent if $A^k = 0$ for some $k$.

    (a) Is $A$ diagonalizable?

    (b) Does $A$ necessarily have a JCF? If so, what does it look like?

16. An $R$-module $P$ is called projective if given any modules $M$ and $N$ with $M \twoheadrightarrow N$ and $f : P \to N$, then there exists a $F$ s.t. the following diagram commutes.

$$\left(\begin{array}{ccc} & & P \\ & {}^F\swarrow & \downarrow_f \\ M & \overset{\phi}{\twoheadrightarrow} & N \end{array}\right), \text{ i.e. } \phi F = f.$$

Prove that this implies that $P$ is a direct summand of a free module.

17. Suppose $R$ is a ring with 1, $L$ is a unitary $R$-module, $M$ and $N$ are submodules of $L$ and both $M + N$ and $M \cap N$ are finitely generated. Show that $M$ and $N$ are finitely generated.