FIRST MIDTERM EXAM, 10/08/04, MATH511A

PLEASE SHOW ALL YOUR WORK
Time: 1:01 - 1:51 (plus possible bonus minutes)
Score: 100 points
Name:

Problem 1=30 points. Decompose the abelian group $\mathrm{Hom}\left(\mathbb{Z}_{72} \oplus \mathbb{Z}_{108}, \mathbb{Z}_{36} \oplus \mathbb{Z}_{216}\right)$ into a direct sum of finite cyclic groups.

Problem 2=25 points=20 points + 5 points.

(a) Show that any group of order 135 is nilpotent.

(b) Use (a) to compute the number of isomorphism classes of groups of order 135.

Problem 3=30 points =2 times 15 points. For a finite group G, let $s_G \in \mathbb{N}$ be the smallest number n such that we have a monomorphism $G \hookrightarrow S_n$.

(a) Compute $s_{\mathbb{Z}_{180}}$.

(b) Let $p \in \mathbb{N}$ be a prime. If G is a finite p-group, show that p divides s_G.

Problem 4=15 points=10 points+5 points. Let $p \in \mathbb{N}$ be a prime and let $n \in \mathbb{N}$. Let G be a finite, abelian p-group of order p^n.

(a) Show that the order of $\operatorname{End}(G)$ divides p^{n^2}.

(b) Find all such groups G with the property that $|\operatorname{End}(G)| = p^{n^2}$.

Practice Questions for Rings

1. Suppose R is a UFD and $r \in R^*$. Show that there are only finitely many $s \in R$ such that $(s) \geq (r)$.

2. If F is a field and $f(x), g(x) \in F[x]$, whos that the least common multiple $lcm(f(x), g(x))$ is a generator for the ideal $(f(x)) \cap (g(x))$.

3. Let $R = M_2(\mathbb{Z})$, the ring of 2×2 matrices over $\mathbb{Z}$, and $M = M_2(2\mathbb{Z})$. Show that M is a maximal ideal in R, and that $R/M \cong M_2(\mathbb{Z}_2)$.

4. Suppose R and S are nontrivial rings with 1 and $\phi : R \to S$ is a homomorphism such that $\phi(1_R) \neq 0$. IF $\phi(1_R) \neq 1_S$, show that $\phi(1_R)$ is a zero-divisor in S. Conclude that if S is an integral domain, then $\phi(1_R) = 1_S$.

5. Suppose R is a finite commutative ring with 1. Show that every prime ideal of R is maximal.

6. Suppose that $S = \mathbb{Q}[x]$ as an additive group, but that the usual multiplication of polynomials is replaced by composition, i.e. $(f \circ g)(x) = f(g(x))$. Show that S is not a ring.

7. If I is an ideal in a ring R, show that $A(I) = \{r \in R : rI = 0\}$ is an indeal in R.

8. If R is a commutative ring with 1 and I is an ideal in R define
$$\sqrt{I} = \left\{r \in R : r^k \in I,\ \text{some } k \in \mathbb{N}\right\}.$$
   (a) Show that $\sqrt{I}$ is an ideal and that $I \subseteq \sqrt{I}$.
   (b) If P is a prime ideal in R with $I \subseteq P$ show that $\sqrt{I} \subseteq P$.
   (c) If $R = \mathbb{Z}$ and $I = (72)$ calculate $\sqrt{I}$.

9. True or False (proof or counterexample)
   (a) If F and K are fields, R is a ring, and $F \subseteq R \subseteq K$, then R is a field.
   (b) If R is a Euclidean domain and S is a nonzero subring then S is Euclidean.
   (c) There is a ring R with 10 elements such that if $a, b \in R$, $a \neq 0$, $b \neq 0$, then $ab \neq 0$.
   (d) Every UFD is a PID.
   (e) If $r \in R$ (commutative ring with 1) then the set $I = \{r \in R : \exists x \neq 0 = rx\}$ is an ideal.
   (f) If $r \in R$ (comm w/1) then $r = \pm 1$.
   (g) If J is a prime ideal then R/J is a field.

(h) $\mathbb{C}[x] / \left(f(x) = x^3 - x\right) \cong \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$

10. Let R be a PID. Let

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$$

be an increasing sequence of ideals in R. Prove that the sequence is everntually constant, i.e. for some n, $I_n = I_{n+1} = I_{n+2} = \cdots$

11. Give an example of an irreducible polynomial of degree 100 in $\mathbb{C}[x, y]$.

    (a) Give an example of an ideal in a commutative ring which is prime but not maximal.

    (b) Prove that if $f : R \to S$ is a homomorphism of commutative rings and $I \subseteq S$ is a prime ideal, then $f^{-1}(I)$ is prime.

12. Show that every nonzero prime ideal in the ring $Z[i]$ of Gaussian integers is maximal.

13. Suppose R is an ID (with 1) having only finitely many ideals. Prove that R is a field. What if R is just a commutative ring, not a domain?

14. Give an example of a prime ideal in $\mathbb{C}[x, y]$. Find one that is prime but not maximal. Give one that is not principal.

15. Let R be a commutative ring. Recall that $r \in R$ is nilpotent if $r^n = 0$ for some $n > 0$ and that the set of all nilpotent elements in R is an ideal. Show that R/N has no nonzero nilpotent elements.

16. Show that the field $\mathbb{C}$ of complex numbers is isomorphic with the subring

$$\left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} : a, b \in \mathbb{R} \right\}$$

of $M_2(\mathbb{R})$.

17. Let R be the addivite abelian $\mathbb{Z} \oplus \mathbb{Z}$ and let R' be the ring $\text{End}(A)$, with multiplication being composition of functions. Show by example that R' is not commutative.

18. Assume that $\gcd(m, n) = 1$. Prove that $\mathbb{Z}/(mn)\mathbb{Z} \cong \mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/m\mathbb{Z}$.

19. Let R be a commutative ring, and let $M \subseteq R$ be an ideal. Prove that R is a local ring with maximal ideal M if and only if every element of R not in M is invertible. (Recall that a ring is local if it has a unique maximal ideal).

SECOND MIDTERM EXAM, 11/15/04, MATH 511A

PLEASE SHOW ALL YOUR WORK
Time: 1:01-1:51 (plus possible bonus minutes)
Score: 100 points
Name:

Problem 1=20 points. Give an example of a commutative ring R with 1 that has precisely 7 ideals.

Problem 2=25 points=10 points +15 points. Let R be a commutative ring with 1 and let I be an ideal of R. Let

$$\sqrt{I} = \left\{ r \in R | r^k \in I \text{ for some } k \in \mathbb{N} \right\}.$$

(a) If P is a prime ideal of R with $I \subseteq P$, then $\sqrt{I} \subseteq P$.

(b) If $R = \mathbb{R}[x, y]$ and $I = \left((x-y)^2 (x+y)^3 \right)$, find $\sqrt{I}$.

Problem 3=35 points=15 points + 20 points.

(a) Solve the system of congruences $x \equiv 1 \pmod{8}$, $x \equiv 5 \pmod{9}$, and $x \equiv 7 \pmod{11}$.

(b) Let $m \in \mathbb{Z}$ be a square free integer. Let P be a prime ideal of R_m that is nonzero. Then P is a maximal ideal of R_m. If $m = -1$, show that the quotient field R_{-1}/P cannot have precisely 7 elements.

Problem 4=20 points. List all irreducible and monic polynomials $f(x) \in \mathbb{Z}_3[x]$ that have degree 3. What is their number?

Final Exam Practice

Math 511a

1 Groups

1. Determine all the homomorphisms from S_3 to A_4.
2. Let G be a group of order pqr, where p, q, r are primes and $p > q > r$. Show that G is solvable.
3. Let G be the group of all $n \times n$ invertible matrices over $\mathbb{R}$, $n \geq 3$. Show that G is not solvable.
4. Find all the composition series of the group $\mathbb{Z}/42\mathbb{Z}$. Verify that they are equivalent.
5. Find a central series $G_0 \subseteq G_1 \subseteq \cdots \subseteq G_n$ in D_4 such that $G_0 = \{1\}$ and $G_n = D_4$.
6. Give an example of a group G such that G is not nilpotent, but G contains a normal subgroup H such that H and G/H are nilpotent.
7. List all normal subgroups of $A_5 \times A_5$.
8. Suppose S is a set and the symmetric group S_4 acts transitively on S. Determine all possibilities for $|S|$.
9. Show that a group of order 48 must have a normal subgroup of order a power of 2.
10. Let G be the group of real 2×2 matrices of determinant 1, and let H be the subgroup of diagonal matrices.

    (a) Find the normalizer of H in G, $N_G(H)$.

    (b) Find the representatives for the cosets in $N_G(H)$.
11. Let p be a prime number. Let $\mathbb{F}_p$ be the field of p elements. Let $G = GL_2(\mathbb{F}_p)$ be the 2×2 invertible matrices with entries in $\mathbb{F}_p$. Let G act on the vector space $V = \mathbb{F}_p \times \mathbb{F}_p$ in the usual way (by matrix multiplication).

    (a) Show that G has exactly 2 orbits on V.

    (b) Compute the order of the stabilizer of $(1, 0)$.

    (c) Use part (b) to compute the order of G.
12. Either give an example of a finite group having its center of prime index or prove that such a group cannot exist.

13. Suppose p is a prime and G is a finite group. A subgroup K of G is called a normal p-complement if $K \lhd G$ and there is a Sylow p-subgroup P such that $K \cap P = 1$ and $KP = G$. Show that if G has a normal p-complement, then it is unique. Give an example.

14. Let H be the subgroup of S_7, the symmetric group of 7 letters, generated by all 3-cycles. Is the permutation (1234) in H? Explain.

15. Give an example or prove that there does not exist a group of order $5!$ acting transitively on a set with 9 elements.

16. What are the conjugacy classes of S_3?

17. Suppose G is a group of order 45 with a normal subgroup P of order 3^2. Show that G is abelian. (Hint: $\text{Aut}\,(P)$ has order 6 or 24 according to whether P is cyclic or elementary abelian).

18. True or false: If G is a nonabelian group then it has abelian subgroups H_α such that $G = \cup_\alpha H_\alpha$ and $\cap_\alpha H_\alpha = 1$.

19. Show that the alternating group A_6 has no subgroup of order 72.

2 Rings

1. Determine positive integers n such that $\mathbb{Z}_n$ has no nonzero nilpotent elements.

2. Write the proof if the statement is true; otherwise give a counterexample

   (a) In a ring R, if a and b are idempotent elements, then $a + b$ is an idempotent element.

   (b) In a ring R, if a and b are nilpotent elements, then $a + b$ is nilpotent.

   (c) Every finite ring with 1 is an integral domain.

   (d) There exists a field with seven elements.

   (e) The characteristic of an infinite ring is always 0.

   (f) An element of a ring R which is idempotent, but not a zero divisor, is the identity element of R.

   (g) If a and b are two zero divisors, then $a + b$ is also a zero divisor in a ring R.

   (h) In a finite field F, $a^2 + b^2 = 0$ implies $a = 0$ or $b = 0$ for all $a, b \in F$.

   (i) In a field F, $(a + b)^{-1} = a^{-1} + b^{-1}$ for all nonzero elements such that $a + b \neq 0$.

   (j) There exists a field with six elements.

3. Let R be a ring such that R has no zero divisors. Show that if every subring of R is an ideal of R, then R is commutative.

4. Prove or give counterexample

   (a) There exist only two homomorphisms from the ring of integers into itself.

   (b) The mapping $f : Z \to Z$ defined by $f(n) = 3n$ is a group homomorphism, but not a ring homomorphism.

   (c) The only isomorphism of a ring R onto itself is the identity mapping of R.

   (d) Let R be a ring with 1. Let $f : R \to S$ be a ring homomorphism. Then $f(1)$ is the identity element of S.

   (e) A nonzero homomorphism from a field into a ring with more than one element is a monomorphism.

   (f) Every nontrivial homomorphic image of an integral domain is an integral domain.

5. An idempotent e of a ring R is called a central idempotent if $e \in C(R)$, the center of the ring and $e^2 = e$. Let R be a ring with 1 and e be a central idempotent in R. Show that

   (a) $1 - e$ is a central idempotent in R;

   (b) eR and $(1-e)R$ are ideals of R;

   (c) $R = eR \oplus (1-e)R$

6. Let R be a commutative ring with 1 and $f(x) = a_0 + a_1 x + ... + a_n x^n \in R[x]$. If a_0 is a unit and $a_1, a_2, ..., a_n$ are nilpotent elements, prove that $f(x)$ is invertible.

7. Let $f(x) = x^6 + x^3 + 1$. Show that $f(x)$ is irreducible over $\mathbb{Q}$.

8. Give an example of a primitive polynomial which has no root in $\mathbb{Q}$ but is reducible over $\mathbb{Z}$.

9. Show that a proper ideal I of a ring R is a maximal ideal if and only if for any ideal A of R either $A \subseteq I$ or $A + I = R$.

10. Let $f(x) = x^5 + 12x^4 + 9x^2 + 6$. Show that the ideal $I = (f(x))$ is maximal in $\mathbb{Z}[x]$.

11. The ring $R = \mathbb{Q}[x] / \langle x^4 - 16 \rangle$ is a direct sum of fields. Describe the fields explicitly and determine how many of each appear as direct summands.

12. Let $f : R \to S$ be a homomorphism of commutative rings. Prove that $I \subset S$ is a prime ideal, then $f^{-1}(I)$ is also a prime ideal. Give an example where I is maximal but $f^{-1}(I)$ is not maximal.

3 Fields

1. Let E be a field extension of the field F with $[E : F] = p$, where p is a prime. Show that for any element $a \in E \backslash K$ we have $E = K(a)$. Hint: Study the subfields of E.

2. (i) Let F be a field and a, b be members of a field containing F. Suppose that a and b are algebraic of degree m and n over F and $(m, n) = 1$. Show that $[F(a, b) : F] = mn$. (ii) Show this is not necessarily true if $(m, n) \neq 1$.

3. Consider the unique factorization domain $F[t]$, where F is a field and t is transcendental over F. Show that the polynomial $x^2 + tx + t \in F(t)[x]$ is irreducible over $F(t)$. Also show that $x^2 + tx + t \in F(x)[t]$ is irreducible over $F(x)$.

4. Find the splitting field for the following polynomials over $\mathbb{Q}$.

   (i) $x^4 + 1$, (ii) $x^6 + x^3 + 1$

5. Find a splitting field S of $x^4 - 10x^2 + 21$ over $\mathbb{Q}$. Find $[S : \mathbb{Q}]$ and a basis for the splitting field over $\mathbb{Q}$.

6. If F is a field with a finite number of element, prove that F is not algebraically closed.

7. Let $f(x) = x^n - 1 \in \mathbb{Q}[x]$. Show that the Galois group of $f(x)$ over $\mathbb{Q}$ is commutative.

8. Find all proper subfields of $\mathbb{Q}\left(\sqrt[3]{2}, \sqrt{3}, i\right)$.

9. Show that the Galois group of $f(x) = x^3 - 5$ over $\mathbb{Q}$ is isomorphic to S_3.

10. Determine the degree of the extension $\mathbb{Q}\left(\sqrt{3 + 2\sqrt{2}}\right)$

FINAL EXAM, 12/17/04, MATH 511A

PLEASE SHOW ALL YOUR WORK
Time: 11:01-1:01 (plus possible bonus minutes)
Score: 200 points
Name:

Problem 1=24 points=8 times 3 points. Define what is:

(a) an infinite nilpotent group N;

(b) a finite Galois field extension $F \hookrightarrow K$;

(c) a right ideal I of a ring S;

(d) a free group $F(X)$ on a set X;

(e) the field of fractions F_R of an integral domain R;

(f) the finite group Q_{10};

(g) the group of units $U(Z)$ of a ring Z;

(h) a simple ring T.

Problem 2=24 points=8 times 3 points. Give examples of:

(a) a field with 121 elements;

(b) a UFD that is not a PID;

(c) an integral domain that has elements that are irreducible but are not prime;

(d) a non-solvable finite group that has order 180;

(e) a ring with unit, that has four elements but is not an integral domain;

(f) a field extension that is not algebraic;

(g) a monoid that is not a group and that is non-abelian;

(h) a field extension of degree p that is not Galois (here p is an arbitrary prime).

Problem 3=22 points=7 points + 15 points. Let R be an integral domain with unit and let $f(x) \in R[x]$ be a monic polynomial. Let P be a prime ideal of R. Let $\bar{f}(x) \in R/P[x]$ be the reduction modulo P of $f(x)$ (i.e. the image of $f(x)$ in $R/P[x]$ via the standard epimorphism $R[x] \twoheadrightarrow R/P[x]$).

(a) Show that if $\bar{f}(x)$ is irreducible, then $f(x)$ is irreducible.

(b) Suppose $R = \mathbb{Q}[t]$ is a polynomial ring over $\mathbb{Q}$ in one variable. Let $f(x) := x^3 + (t-3)x^2 - tx + \left(t^2 - 5t + 9\right) \in R[x]$. Use part (a) to show that $f(x)$ is irreducible.

Problem 4=30 points=3 times 10 points. Let F be a finite field. Let $q := |F|$. For $d \in \{1, 2, 3\}$, let $n_d \in \mathbb{N}$ be the number of monic irreducible polynomials $f(x) \in F[x]$ that are of degree d. Find a formula for n_d in terms of q. Hint: use the fact that $F[x]$ is a UFD.

Problem 5=25 points. Let p and q be two primes such that $p > q$. Show that any finite group of order pq^2 is solvable.

Problem 6=25 points. Let $p \in \mathbb{N}$ be an odd prime. Find the number of subgroups of D_{p^2}. How many of them are normal?

Problem 7=25 points. Let $p \in \mathbb{N}$ be a prime. Find the number of subfields of the field $\mathbb{F}_{p^{36}}$ with p^{36} elements. For each subfield F, compute the Galois group $Gal\left(\mathbb{F}_{p^{36}}/F\right)$.

Problem 8=25 points. Find the splitting field of the following polynomial $x^{20} + x^{15} + x^{10} + x^5 + 1 \in \mathbb{Q}[x]$.