# January 2006
# Algebra Qualifying Exam
Solutions

1A) Suppose $A$ is an $n \times n$ matrix over a field $F$ and that $A^k = 0$ for some $k \in \mathbb{N}$. Show that $A^n = 0$.
**Answer:** Assume that $A^k = 0$.

1B) Let $V$ be a vector space over a field $K$. Show that the following two statements are equivalent:
(a) we have $\dim_K V < \infty$;
(b) any $K$-linear endomorphism $f : V \to V$ satisfies an equation of the form $f^m + a_{m-1}f^{m-1} + \cdots + a_1 f + a_0$, where $m \in \mathbb{N}$ and $a_0, ..., a_{m-1} \in K$.
**Answer:** ()

2A) Let $G$ be a finite group of order $2^n(2m+1)$. Let $H := Hom(G, \mathbb{Z}/2\mathbb{Z})$. Show that $H$ is an abelian group whose order is $2^s$ for some number $s \in \{0, 1, ..., n\}$. Give examples when $s = 0$ and $s = n$.
**Answer:** ()

2B) If $G$ is a group of order $4n + 2$ show that $G$ has a subgroup $H$ of index 2.
**Answer:** ()

3A) If $R$ is a ring show that all non-zero divisors in $R$ have the same additive order. What are the possible orders?
**Answer:** FF

3B) Let $K$ be a field. Let $R = K[x, y]$ be the polynomial ring in two variables with coefficients in $K$. For $a \in K$, let $(x^2, y + ax)$ be the ideal of $R$ generated by $x^2$ and $y + ax$. We consider the ideal $I := \cap_{a \in K} (x^2, y + ax)$ of $R$. Compute the smallest number $n \in \mathbb{N}$ such that $I$ is generated by $n$ elements. Give an example of $n$ elements $i_1, ..., i_n \in I$ that generate $I$.
**Answer:** ()

4A) Let $K$ be a Galois extension of $\mathbb{Q}$ of degree 4. Let $D$ be the set of those square free integers $d \in \mathbb{Z} \setminus \{1\}$ which have the property that $\mathbb{Q}\left(\sqrt{d}\right) \subseteq K$.
(a) Show that the set $D$ has either 1 or 3 elements.
(b) If $D = \{d_1, d_2, d_3\}$ has three elements, then show that the product $d_1 d_2 d_3$ is a perfect square.
(c) If $K = \mathbb{Q}(\zeta_5)$, find $D$.
**Answer:** ()

4B) Determine a Galois closure $K$ for $L = \mathbb{Q}\left(3i - \sqrt{2}\right)$ over $\mathbb{Q}$ and determine the Galois group $G = G(K : \mathbb{Q})$.
**Answer:** Consider $a = 3i - \sqrt{2}$. Then $a - 3i = -\sqrt{2}$ and thus $(a - 3i)^2 = 2$ implies that $a^2 - 6ai - 9 = 2$ and so $a^2 - 11 = 6ai$ and so $(a^2 - 11)^2 = -36a^2$ and so $a^4 - 22a^2 + 121 + 36a^2 = 0$. Thus $a^4 + 14a^2 + 121$ is the minimal polynomial. It is a degree 4 irreducible polynomial and thus $G \leq S_4$. ...

5A) An abelian group has generators $a, b, c, d$ and defining relations $3b + 2c + 8d = 0$, $5a + b - 4c + 8d = 0$, $-2a + b + 4c - 8d = 0$ and $-a + 3b + 2c + 8d = 0$. Express the group as a direct sum of cyclic groups.

**Answer:** Consider the following matrix and its SNF:

$$\begin{bmatrix} 0 & 3 & 2 & 8 \\ 5 & 1 & -4 & 8 \\ -2 & 1 & 4 & -8 \\ -1 & 3 & 2 & 8 \end{bmatrix} \to \begin{bmatrix} -1 & 3 & 2 & 8 \\ 5 & 1 & -4 & 8 \\ -2 & 1 & 4 & -8 \\ 0 & 3 & 2 & 8 \end{bmatrix} \to$$

$$\begin{bmatrix} -1 & 3 & 2 & 8 \\ 0 & 16 & 6 & 48 \\ 0 & -5 & 0 & -24 \\ 0 & 3 & 2 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 16 & 6 & 48 \\ 0 & -5 & 0 & -24 \\ 0 & 3 & 2 & 8 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 2 & 8 \\ 0 & -5 & 0 & -24 \\ 0 & 16 & 6 & 48 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 2 & 8 \\ 0 & -2 & 2 & -16 \\ 0 & 1 & -4 & 8 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -4 & 8 \\ 0 & -2 & 2 & -16 \\ 0 & 3 & 2 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & -6 & 0 \\ 0 & 0 & 14 & -16 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -6 & 0 \\ 0 & 0 & 14 & -16 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -6 & 0 \\ 0 & 0 & 2 & -16 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & -16 \\ 0 & 0 & -6 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & -16 \\ 0 & 0 & 0 & -48 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 48 \end{bmatrix}$$

Thus $G \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/48\mathbb{Z}$.

5B) Let $K = \mathbb{F}_5 = \{[0], [1], [2], [3], [4]\}$. Let $V := K^3$. We consider the four vectors $v_1 = ([2], [3], [1])$, $v_2 = ([1], [2], [4])$, $v_3 = ([0], [1], [1])$ and $v_4 = ([4], [2], [1])$. Show that $B = \{v_1, v_2, v_3\}$ is a $K$-basis of $V$. Compute the coordinates of $v_4$ with respect to $B$.

**Answer:** ()