January 2000
Algebra Qualifying Exam
Sample Solutions

1A) Let V be a finite dimensional vector space over a field F, and let U and W be subspaces of V.

(a) Show that $V \neq U \cup W$ unless $V = U$ or $V = W$.

(b) If $\dim(U) = \dim(W)$ show that U and W have a common complement, i.e. there is a subspace X so that $V = U \oplus X = W \oplus X$. (Hint: You may wish to use part (a)).

Answer: (a) Let $V = U \cup W$. Let $x \in U$ and $y \in W$. Then $x + y \in U \cup W$. So $x + y \in U$ or $x + y \in W$. So $x \in W$ or $y \in U$. This implies that $U \subseteq W$ or $W \subseteq U$ and thus $U \cup W = U$ or W. And as $V = U \cup W$ we have that $V = U$ or $V = W$.

(b) Assume $\dim(V) = n$ and $\dim(U) = \dim(W) = m$. Take a basis for U and W, $\{u_1, ..., u_m\}$ and $\{w_1, ..., w_m\}$ resprectively. We will now add a $u_{m+1} \notin U$ to the basis and also show that we can pick it as such so that $u_{m+1} \notin W$. If there is no such u_{m+1} then $U \cup W = V$ and thus $V = W$ or $U = V$, a contradiction. So now let $U_1 = U \oplus span_F \{u_{m+1}\}$ and $W = W \oplus span_F \{u_{m+1}\}$. Now continue this process until $\dim_F U_k = \dim V$ and thus we take $\{u_{m+1}, ..., u_n\} = X$.

1B) Let F be a field and V an n-dimensional vector space over F. There is an F-linear endomorphism T of the tensor product $V \otimes V$ mapping $v \otimes w$ to $T(v \otimes w) = w \otimes v$ for all $v, w \in V$. Determine the eigenvalues of T and furthermore determine bases for corresponding eigenspaces.

Answer: For V choose a basis $\{v_1, ..., v_n\}$. So choose basis for $V \otimes V$ as $\{v_i \otimes v_j : 1 \leq i, j \leq n\}$. Order the basis as follows:

$$\{v_1 \otimes v_1, ..., v_n \otimes v_n, v_1 \otimes v_2, v_2 \otimes v_1, ..., v_1 \otimes v_n, v_n \otimes v_1, ..., v_{n-1} \otimes v_n, v_n \otimes v_{n-1}\}$$

Now we can explicity describe the linear transformation with the following matrix:

$$\begin{pmatrix} I_n & & & & & & & & \\ & 0 & 1 & & & & & & \\ & 1 & 0 & & & & & & \\ & & & 0 & 1 & & & & \\ & & & 1 & 0 & & & & \\ & & & & & \ddots & & & \\ & & & & & & \ddots & & \\ & & & & & & & 0 & 1 \\ & & & & & & & 1 & 0 \end{pmatrix}$$

The determinant of a block diagonal matrix is computed as the product of the determinants of the blocks. Thus $\det(T - xI) = (1-x)^n (x^2 - 1)^{(n^2 - n)/2}$. So we have eigenvalues of 1 with multiplicity $n + (n^2 - n)/2$ and -1 with multiplicity $(n^2 - n)/2$. The eigenspace corresponding to 1 has basis

$$\left\{ \begin{array}{c} v_i \otimes v_i : 1 \leq i \leq n \\ v_i \otimes v_j + v_j \otimes v_i : 1 \leq i < j \leq \end{array} \right\}$$

and the eigenspace corresponding to -1 has basis $\{v_i \otimes v_j - v_j \otimes v_i : 1 \leq i < j \leq n\}$.

2A) Suppose G is a group, $H \leq G$, and $x^2 \in H$ for all $x \in G$. Show that $H \lhd G$ and G/H is abelian.
Answer: If a subgroup contains the commutator subgroup then it is normal so we need to show that $G' \subseteq H$. Let $x^{-1}y^{-1}xy \in G'$. As

$$\begin{aligned} x^{-1}y^{-1}xy &= x^{-1}y^{-1}x^{-1}y^{-1}yxxyxxx^{-1}x^{-1} = \\ &= \left(x^{-1}y^{-1}\right)^2 (yxx)^2 \left(x^{-1}\right)^2 \in H \end{aligned}$$

so $G' \subseteq H$ and so $H \lhd G$. As $(a^{-1}b^{-1}ab)\, H = H$, $(aH)(bH) = (bH)(aH)$.

2B) Let G be a finite group, $K \lhd G$ and P a Sylow p-subgroup of K for some prime p. Show that $G = N_G(P)\, K$.

3A) Suppose R is a principal ideal domain (PID) and $I \neq 0$ is an ideal in R. Show that the set $\{J : J \text{ is an ideal in } R \text{ and } I \subseteq J\}$ is finite.

3B) Let K be a field and let $K[x_1, ..., x_n]$ be the polynomial ring in n indeterminates. Let P be a minimal prime ideal (i.e. a nonzero prime ideal that contains no smaller nonzero prime ideal). Show that P is a principal ideal generated by an irreducible polynomial f.

4A) If $\alpha = \sqrt{3 + \sqrt{15}} \in R$ and set $F = \mathbb{Q}(\alpha)$. Show that F is *not* a Galois extension of $\mathbb{Q}$. Let $K \subseteq \mathbb{C}$ be the Galois closure of F. Determine K explicitly, and in particular find $[K : \mathbb{Q}]$.

4B) Suppose that $f(x) \in \mathbb{Q}[x]$ is irreducible of degree 4. Show that the Galois group of $f(x)$ cannot be the quaternion group $\mathcal{Q}$ of order 8.

5A) Suppose R is a ring with 1, L is a unitary (left) R-module, M and N are submodules of L and both $M + N$ and $M \cap N$ are finitely generated. Show that M and N are finitely generated.

5B) Let T be the $\mathbb{Z}[i]$-module homomorphism from $\mathbb{Z}[i]^2$ to $\mathbb{Z}[i]^2$ defined by the matrix

$$\begin{pmatrix} 2i & 4i+2 \\ 2i-2 & i \end{pmatrix}$$

Determine whether T is one-to-one and whether T is onto.