January 1996
Algebra Qualifying Exams
Solutions

1A) Let V be a vector space of dimension n over $\mathbb{R}$ and let $0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_{n-1} \subseteq V_n = V$ be a sequence of subspaces with $\dim V_i = i$ for all i. Define a group G by

$$G = \{g \in GL(V) : gV_i \subseteq V_i \text{ for all } i\}$$

Give as complete a description as possible of the structure of G.
Answer: Consider any k dimensional subspace V_k. So we may choose a standard basis (or change basis if already given one) such that we write

$$V_k = \begin{pmatrix} b_1 \\ \vdots \\ b_k \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Thus we want

$$\begin{pmatrix} a_{11} & \cdots & & a_{1n} \\ \vdots & \ddots & & \\ & & & \\ a_{n1} & & & a_{nn} \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_k \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} c_1 \\ \vdots \\ c_k \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

This is only possible if G is the set of upper triangular matrices.

1B) Show that every 2×2 real matrix with all positive entries can be diagonalized over $\mathbb{R}$.
Answer: A matrix is diagonalizable if and only if its minimal polynomial splits into distinct linear factors. Thus it suffices to show that its characteristic polynomial does so as a minimal polynomial always divides the characteristic polynomial. The characteristic polynomial of an arbitrary matrix $\{a_{ij}\} = A$ is $\det(A - xI) = x^2 - (a_{11} + a_{22})x + a_{11}a_{22} - a_{21}a_{12}$. This has two distinct roots if and only if $(a_{11} + a_{22})^2 - 4(1)(a_{11}a_{22} - a_{21}a_{12}) > 0$. We rewrite this as

$$a_{11}^2 + a_{22}^2 + 2a_{11}a_{22} - 4(1)(a_{11}a_{22} - a_{21}a_{12}) = (a_{11} - a_{22})^2 + 4a_{21}a_{12}$$

which has $(a_{11} - a_{22}) \geq 0$ and $4a_{21}a_{12} > 0$ so the sum is > 0 and thus has distinct roots, thus distinct eigenvalues, thus diagonalizable.

2A) Find all groups of order 33.
Answer: As $33 = 3 \cdot 11 = pq$ where p and q are distinct primes and $p \neq 1 \bmod q$ and $q \neq 1 \bmod p$ tell us that there is a unique Sylow 3-subgroup P and a unique Sylow 11-subgroup Q. Each of these is thus normal. Also P and Q are cyclic as they are of prime order. So say $P = \langle x \rangle$ and $Q = \langle y \rangle$. The $x^{-1}y^{-1}xy \in P \cap Q$ as each is normal and $P \cap Q = 1$. So $xy = yx$ and so $|\langle xy \rangle| = 33 = 3 \cdot 11$ and so $G = \langle xy \rangle$ is a cyclic group and thus the only group of order 33 is cyclic.

2B) Either give an example of a finite group having its center of prime index or prove that such a group cannot exist.
Answer: Proceed by contradiction. Assume a group G has center $Z(G)$ such that $[G : Z(G)] = p$, a prime. Then as $Z(G) \lhd G$, $G/Z(G)$ has order p and is thus cyclic. This implies that G is abelian. But then $[G : Z(G)] = 1$, a contradiction. So it is not possible.

3A) The ring $R = \mathbb{Q}[x] / \langle x^4 - 16 \rangle$ is a direct sum of fields. Describe the fields explicitly and determine how many of each appears as direct summands.

Answer: By the chinese remainder theorem, if R is a ring with 1 and $I_j + I_k = R$ for all $j \neq k$ then $R/\langle I_1 \cap \cdots \cap I_n \rangle \cong R/I_1 \oplus \cdots \oplus R/I_n$. We first note that $\mathbb{Q}[x]$ is a PID. And so

$$\left\langle x^4 - 16 \right\rangle = \left(\left\langle x^2 + 4 \right\rangle \cap \langle x - 2 \rangle \cap \langle x + 2 \rangle \right)$$

2nd we check the conditions $I_j + I_k = R$.

$x^2 + 4 - (x-2)(x+2) = 8 \in U(\mathbb{Q})$ and so $\left\langle x^2+4 \right\rangle + \langle x+2 \rangle = R$

$x^2 + 4 - (x+2)(x-2) = 8 \in U(\mathbb{Q})$ and so $\left\langle x^2+4 \right\rangle + \langle x-2 \rangle = R$

$(x+2) - (x-2) = 4 \in U(\mathbb{Q})$ and so $\langle x+2 \rangle + \langle x-2 \rangle = R$.

So

$$\begin{aligned} \mathbb{Q}[x] / \left\langle x^4 - 16 \right\rangle &\cong \mathbb{Q}[x] / \left\langle x^2 + 4 \right\rangle \oplus \mathbb{Q}[x] / \langle x - 2 \rangle \oplus \mathbb{Q}[x] / \langle x + 2 \rangle \\ &\cong \mathbb{Q}[x] / \left\langle x^2 + 4 \right\rangle \oplus \mathbb{Q} \oplus \mathbb{Q} \end{aligned}$$

The last thing we do is to show that $\mathbb{Q}[x] / \left\langle x^2 + 4 \right\rangle \cong \mathbb{Q}(i)$. Consider the map

$$\begin{array}{ccc} \mathbb{Q}[x] & \longrightarrow & \mathbb{Q}(i) \text{ by} \\ f(x) & \mapsto & f(2i) \end{array}$$

This map can easily be checked to be onto. The kernel of the map is $\left\langle x^2 + 4 \right\rangle$. Thus we have our isomorphism by the FHT. Thus the final answer is

$$\mathbb{Q}[x] / \left\langle x^4 - 16 \right\rangle \cong \mathbb{Q}(i) \oplus \mathbb{Q}^2$$

We also knew that all were going to be fields as all were prime ideals and in a PID we know that prime if and only if maximal and an ideal I is maximal if and only if R/I is a field.

3B) (i) Proof or counterexample: Every UFD is a PID.

(ii) Give an explicit example (with justification) of an irreducible polynomial of degree 100 in $\mathbb{C}[x, y]$.

Answer: (i) False, any UFD is also a UFD over an amount of indeterminates. So look at $\mathbb{Z}[x]$ and $\mathbb{R}[x, y]$ where R is a UFD. In $\mathbb{Z}[x]$ the ideal $\langle 2, x \rangle$ is not principal and $\langle x, y \rangle$ is not principal in $\mathbb{R}[x, y]$.

(ii) Take $y^{100} - x$. This is irreducible if we look at the Eisenstein criterion by consider the polynomial ring $\mathbb{C}[x, y] = \mathbb{C}[x][y]$. Thus the ring for Eisenstein is $\mathbb{C}[x]$. The Eisenstein criterion states that: Suppose R is a PID (which $\mathbb{C}[x]$ is) and $f(x) = a_0 + a_1 x + ... + a_n x^n$ is primitive (i.e. has content $= 1 = GCD(a_0, ..., a_n)$. Suppose there is a prime $p \in R$ such that $p | a_i$ for $0 \leq i \leq n-1$, but $p \nmid a_n$ and $p^2 \nmid a_0$. Then $f(x)$ is irreducible. As x is prime in $\mathbb{C}[x]$ we have satisfied the Eisenstein criterion.

4A) Find the Galois group over $\mathbb{Q}$ of $f(x) = x^5 - 80x + 2$. (Hint: How many real roots does $f(x)$ have?).

Answer: There are 3 real roots and two complex nonreal roots as $f'(x) = 5x^4 - 80$ and $f''(x) = 20x^3$ tells us that we have turning points (local max/min) as $5x^4 - 80 = 0 \implies x^4 = 16$ and so $x = \pm 2$ are critical points with $(2, -126)$ a local min and $(-2, 130)$ a local max. So there are 3 real roots and two strictly complex which must be conjugate. As $f(x)$ is irreducible by Eisenstein with $p = 2$, $f(x)$ splits in some K with $[K : \mathbb{Q}] \geq 5$ and $G(K : \mathbb{Q}) \leq S_5$. As there are 2 strictly complex roots and 3 real we must have a transitive action and so we have a 5 cycle and a two cycle. And this generates S_5.

4B) Let $f(x) = x^4 + 5x^2 + 9 \in \mathbb{Q}[x]$. Is $f(x)$ irreducible? Determine its Galois group over $\mathbb{Q}$.

Answer: $f(x) = \left(x^2 + x + 3\right)\left(x^2 - x + 3\right)$ and so f is reducible. It has roots

$$\frac{\pm 1 \pm \sqrt{-11}}{2}$$

and thus a splitting field of $K = \mathbb{Q}\left(\sqrt{11} i\right)$ and as this is a degree 2 extension we have that $G \cong \mathbb{Z}_2$.

5A) Proof or counterexample: Every submodule of $\mathbb{Z}$ as a $\mathbb{Z}$-module is free.

Answer: Proof: Every submodule of $\mathbb{Z}$ as a $\mathbb{Z}$-module is an ideal in the ring $\mathbb{Z}$ as a $\mathbb{Z}$-module. The only ideals are $n\mathbb{Z}$ and as $n\mathbb{Z} \cong_{\mathbb{Z}} \mathbb{Z}$ we have that every submodule is free. Another way to see this is to note that $\mathbb{Z}$ is a PID and note that every submodule S of a free module R is free with $rank(S) \leq rank(R)$ by Grove Theorem 2.9 on page 133.

5B) Let A be the abelian group with presentation:

$$A = \langle a, b, c : 2a + 4b + 2c = 2a + 10b + 8c = 0 \rangle$$

Determine the order and structure of the torsion subgroup T of A.

Answer: We quote the following theorem: Suppose R is a PID, M and N free R-modules of finite rank and $f \in Hom_R(M, N)$. Set $E = \mathrm{Im}(f)$. Suppose there are bases $\{x_1, ..., x_n\}$ for M and $\{y_1, ..., y_m\}$ for N such that the matrix representing f relative to $\{x_i\}$ and $\{y_j\}$ has the block diagonal form

$$B = \begin{pmatrix} U & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where $U = \begin{pmatrix} u_1 & & 0 \\ & \ddots & \\ 0 & & u_s \end{pmatrix}$ and $D = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_k \end{pmatrix}$, with $u_i \in U(R)$ all i, $0 \neq d_j \notin U(R)$, all j and $d_j | d_{j+1}$ for $1 \leq j \leq k-1$. Then the quotient module N/E is the direct sum of cyclic submodules $R\langle y_i + E \rangle$, $s + 1 \leq i \leq m$, its torsion submodule has invariant factors $d_k, ..., d_{k-1}, .., .d_1$ and its rank is $m - s - k$.

Or a simple way of putting it, put the relations in a matrix, compute the SNF and write down the answer:

$$\begin{pmatrix} 2 & 4 & 2 \\ 2 & 10 & 8 \end{pmatrix} \to \begin{pmatrix} 2 & 4 & 2 \\ 0 & 6 & 6 \end{pmatrix} \to \begin{pmatrix} 2 & 0 & 0 \\ 0 & 6 & 6 \end{pmatrix} \to \begin{pmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \end{pmatrix}$$

and thus the torsion subgroup has order 12 and the group is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_6 \oplus \mathbb{Z}$.