January 1994
Algebra Qualifying Exam
Sample Solutions

1A) Find all (real) c such that the system

$$\begin{aligned} 2x + c(c-1)y + 3z &= c+1 \\ -2x + c(c-1)y - 3z &= c-1 \end{aligned}$$

has a solution, and find the dimension of the space of solutions if they exist.
Answer: By adding the two equations we get that $2c(c-1) = 2c$. If $c = 0$ then $y = y$, $x = (1-3z)/2$, $z = z$ and so $(x, y, z) = y(0, 1, 0) + (1/2, 0, 0) + z(-3/2, 0, 1)$. This is a 2-dimensional solution space. If $c \neq 0$ then $(c-1)y = 1$. If $c = 1$ then we have no solution as we get $2 = 0$. So if $c \neq 0$, then $y = 1/(c-1)$. So $2x + c + 3z = c + 1$ and $-2x + c - 3z = c - 1$. Therefore we only need satisfy $2x + 3z = 1$ and so this is a 1-dimensional solution space.

1B) Find the characteristic polynomial, minimal polynomial, rational canonical form, and Jordan canonical form of

$$A = \begin{bmatrix} 0 & 4 & 0 \\ 2 & 0 & 8 \\ 0 & -1 & 0 \end{bmatrix}$$

Answer: We compute $Det(A - xI)$ and get $Det(A - xI) = -x^3$. Thus the characteristic polynomial is $-x^3$. The minimal polynomial must divide x^3 and so is either x, x^2, or x^3. We check and see that $A \neq 0$ and $A^2 \neq 0$ and so the minimal polynomial is also x^3. Thus we only have one invariant factor x^3 and one elementary divisor x^3. Thus the RCF and JCF are the same and we have $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

2A) Suppose S is a set and the symmetric group S_4 acts transitively on S. Determine all possibilities for $|S|$.
Answer: There is a bijection between all possible actions of a group G on a set A with possible homomorphisms from G to S_A. By the orbit stabilizer theorem, we know that $|Orbit_G(s)| = [G : Stab_G(s)]$. For a transitive action the size of the entire set occurs as the index of a subgroup of G. This gives us a list of possible set sizes. The most canonical map to use is $G \times G/H \to G/H$ by $g_1 \times g_2H \to g_1g_2H$ and this is a well-defined action and we can identify each of the elements of our given set with one of the cosets. This is a transitive actions as if we want to move g_2H to g_3H we simply act by $g_3g_2^{-1}$. So we must find all possible subgroups of S_4. They have size $1, 2, 3, 4, 6, 8, 12, 24$ and so we have corresponding indices $24, 12, 8, 6, 4, 3, 2, 1$. So we could have the size of S as all possible divisors of 24.

2B) If $p \in \mathbb{Z}$ is a prime, determine all groups of order $2p$.
Answer: If $p = 2$ we know that the only possibilities are $\mathbb{Z}_4$ and $V_4 = \mathbb{Z}_2 \times \mathbb{Z}_2$. If $p \neq 2$ then $p \equiv 1 \bmod 2$. Choose elements x and y in G of orders p and 2 respectively and set $P = \langle x \rangle$ and $Q = \langle y \rangle$. As $p > 2$, the Sylow p-subgroup is unique and so $P \lhd G$ and $[G : N_G(Q)] = 1$ or p. If $[G : N_G(Q)] = 1$ then $G = \langle xy \rangle$ is cyclic and $G \cong \mathbb{Z}_p$. On the other hand suppose that $[G : N_G(Q)] = p$. Then $y^{-1}xy = x^n$ for some integers $2 \leq n < p$. So $x^{-1}xy^k = x^{n^k}$ if $0 < k \in \mathbb{Z}$ and in particular $y^{-2}xy^2 = x^{n^2}$ and so $n^2 \equiv 1 \bmod p$. So G has generators x and y that satisfy the relation $x^p = y^2 = 1$ and $y^{-1}xy = x^n$ for $2 \leq n < p$. This is the presentation of a dihedral group of order $2p$. So the only possibilities are $\mathbb{Z}_{2p}$ or D_p or V_4.

3A) Suppose R is an ID (with 1) having only finitely many ideals. Prove that R is a field. What if R is just a commutative ring, not a domain?
Answer: As there are only a finite number of ideals, we know that R is artinian, i.e. satisfies the Descending Chain Condition (DCC). Let $0 \neq a \in R$. Then

$$\langle a \rangle \supseteq \langle a^2 \rangle \supseteq \cdots \supseteq \langle a^{n-1} \rangle \supseteq \langle a^n \rangle$$

where each $\langle a^i \rangle$ is an ideal. As R is an integral domain (ID) and $a \neq 0$ we know that $a^i \neq 0$. Artinian implies that there exists an n such that $\langle a^n \rangle = \langle a^{n+1} \rangle = ...$ and so $a^n \in \langle a^{n+1} \rangle$. So there is an $r \in R$ such that $a^n = ra^{n+1}$ and so $1 = ra$ and so R *is* a field.

Without being an integral, it is definitely not true. Take $\mathbb{Z}_4$ for example as 2 has no multiplicative inverse as it is a zero divisor.

3B) Describe all semisimple rings having $10,000$ elements.
Answer: First note that any finite division ring is a field by a theorem of Weddeburn. Also note that any finite ring is Artinian. Thus we have a semisimple Artinian ring and thus by the Weddeburn-Artin structure theorem we have that R is isomorphic to a direct sum of full matrix algebras over a division ring. We factor $10,000$ for possible sizes

$$10,000 = 2^4 5^4$$

The possibilities for the first term 2^4 is

$$A = M_2(\mathbb{F}_2), \mathbb{F}_{2^4}, \mathbb{F}_{2^3} \times \mathbb{F}_2,\ \mathbb{F}_{2^2} \times \mathbb{F}_{2^2},\ \mathbb{F}_{2^2} \times \mathbb{F}_2 \times \mathbb{F}_2, \text{ or } \mathbb{F}_2 \times \mathbb{F}_2 \times \mathbb{F}_2 \times \mathbb{F}_2$$

The possibilities for the second term 5^4 is

$$B = M_2(\mathbb{F}_5), \mathbb{F}_{5^4}, \mathbb{F}_{5^3} \times \mathbb{F}_5,\ \mathbb{F}_{5^2} \times \mathbb{F}_{5^2},\ \mathbb{F}_{5^2} \times \mathbb{F}_5 \times \mathbb{F}_5, \text{ or } \mathbb{F}_5 \times \mathbb{F}_5 \times \mathbb{F}_5 \times \mathbb{F}_5$$

Thus all possibilities are $A \times B$. There are a total of 36 possibilities.

4A) Suppose F, K, and L are fields with $F \subseteq K \subseteq L$ and $[L : F]$ finite. Either prove or give a counterexample for each of the following 3 assertions.
(a) If L is Galois over F then L is Galois over K.
(b) If L is Galois over F then K is Galois over F.
(c) If L is Galois over K and K is Galois over F then L is Galois over F.
Answer: (a) By the FTGT we know there is a 1-1 inclusion reversing correspondence between subgroups of the Galois group G, $|G| = |Aut(F/L)| = n < \infty$ and the subfields of the extension $L : F$. So there is a subgroup $H \leq G$ such that H is the group of automorphisms of L with K as its fixed field. $|Aut(L/K)| = |H| = [L : K]$ so it is Galois.

(b) False take the extension $\mathbb{Q}\left(2^{1/4}, i\right) = L$ and $K = \mathbb{Q}\left(2^{1/4}\right)$ (not a normal extension as minimal polynomial $x^4 - 2$ does not split and has root $2^{1/4}$, we are missing non-real complex roots).

(c) Take $F = \mathbb{Q}$, $K = \mathbb{Q}\left(2^{1/2}\right)$, and $L = \mathbb{Q}\left(2^{1/4}\right)$. As both $[K : F]$ and $[L : K]$ are degree 2 extensions they must be normal. And as $charQ = 0$ we know that they are separable extensions. Thus they are Galois extensions. However $\mathbb{Q}\left(2^{1/4}\right)$ is not Galois as the minimal polynomial $x^4 - 2$ has 2 strictly complex roots.

4B) Suppose F and K are fields with $F \subseteq K$ and $a \in K$ is algebraic over F with $[F(a) : F]$ odd. Show that $F\left(a^2\right) = F(a)$.
Answer: Let $f(x)$ split in K' and $g(x)$ in K, where $g(x) = f\left(x^2\right)$. Then clearly $K' \leq K$ as if α_i are the roots of $f(x)$ then $\pm\sqrt{\alpha_i}$ are the roots for $g(x)$. Thus the splitting field for $g(x)$ must be a degree 2^n extension of K' for some $n \geq 0$. We know that

$$\begin{aligned} [K : \mathbb{Q}] &= [K : K'][K' : \mathbb{Q}] \\ \text{Odd} &= 2^n \cdot x \end{aligned}$$

And this is possible if and only if $n = 0$ and thus $K = K'$.

5A) Let M be the $\mathbb{Z}$-module $\mathbb{Z}\oplus(\mathbb{Z}/3\mathbb{Z})$. Give a precise and explicit description of the ring $End_{\mathbb{Z}}(M)$.
Answer: First we note that

$$\begin{aligned} End(\mathbb{Z}\oplus(\mathbb{Z}/3\mathbb{Z})) &= Hom(\mathbb{Z}\oplus(\mathbb{Z}/3\mathbb{Z}), \mathbb{Z}\oplus(\mathbb{Z}/3\mathbb{Z})) \\ &\cong End(\mathbb{Z}) \oplus Hom(\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}) \oplus Hom(\mathbb{Z}/3\mathbb{Z}, \mathbb{Z}) \oplus End(\mathbb{Z}\mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}) \\ &\cong \mathbb{Z} \oplus \mathbb{Z}_3 \oplus \{0\} \oplus \mathbb{Z}_3 \end{aligned}$$

5B) Suppose A and B are finite abelian groups each having all Sylow subgroups cyclic; view A and B as $\mathbb{Z}$-modules. Calculate $A \otimes_{\mathbb{Z}} B$ and determine its Sylow subgroups.

Answer: The fundamental theorem of finitely generated abelian groups and the fact that abelian implies all subgroups normal implies Sylow subgroups are unique and thus

$$\begin{aligned} A &\cong \mathbb{Z}_{p_1^{e_1}} \oplus \cdots \oplus \mathbb{Z}_{p_k^{e_k}} \\ B &\cong \mathbb{Z}_{q_1^{f_1}} \oplus \cdots \oplus \mathbb{Z}_{q_l^{f_l}} \end{aligned}$$

We now use the fact that we can distribute direct sums over tensors and also that $\mathbb{Z}_r \otimes \mathbb{Z}_s \cong \mathbb{Z}_{(r,s)}$ where (r, s) is the gcd. So for each $\mathbb{Z}_{p_i^{e_i}} \otimes \mathbb{Z}_{q_j^{f_j}}$ we have

$$\mathbb{Z}_{p_i^{e_i}} \otimes \mathbb{Z}_{q_j^{f_j}} \cong \left\{ \begin{array}{c} 0,\, p_i \neq q_j \\ \mathbb{Z}_{p_i^{\min(e_i, f_j)}},\, p_i = q_j \end{array} \right\}$$

Let $A = \{p_1, ..., p_k\}$ and $B = \{q_1, ..., q_l\}$. Then $A \cap B = \{p_{i_1}, ..., p_{i_m}\}$. In the tensor product we will have that the corresponding powers of the primes are e_{i_1}, the minimum of the prime powers of the intersecting $p_i^{e_i}$ and $q_j^{f_j}$. Thus the Sylow subgroups have order $p_{i_p}^{e_p}$.