August 1995
Algebra Qualifying Exams
Sample Solutions

1A) Let $M = \begin{bmatrix} -12 & 18 \\ -8 & 12 \end{bmatrix}$. Find all eigenvalues of M and M^{100}.

Answer: $\det(M - xI) = x^2$. So the eigenvalues are 0 with corresponding eigenvector $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$. As $M^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and so $M^{100} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. This also has eigenvalues 0 with multiplicity 2.

1B) Let V be a finite dimensional vector space over a field F and let $(\, , \,) : V \times V \to F$ be a bilinear form. Prove that

$$\dim_F \{v \in V : (v,w) = 0 \text{ for all } w \in V\} = \dim_F \{w \in V : (v,w) = 0 \text{ for all } v \in V\}.$$

Answer: $(\, , \,) : V \times V \to F$ so we know that we can write $(u,v) = u \cdot A \cdot v^T$ for some matrix A.

$$\begin{aligned} \{v \in V : (v,w) = 0 \text{ for all } w \in V\} &= \left\{v \in V : v \cdot A \cdot w^T = 0 \text{ all } w \in V\right\} \\ \{w \in V : (v,w) = 0 \text{ for all } v \in V\} &= \left\{w \in V : v \cdot A \cdot w^T = 0 \text{ all } v \in V\right\} \end{aligned}$$

Now we consider possible dimensions

$$\begin{aligned} \dim (u \in V : u \cdot A = 0) &= \dim Null\left(A^T\right), \text{ so } \left(A^T u^T\right)^T = 0 \\ \dim \left(w \in V : A \cdot w^T = 0\right) &= \dim Null\,(A) \end{aligned}$$

Also,

$$\begin{aligned} \dim Null\,(A) + Rank A &= n = \dim V \\ \dim Null\left(A^T\right) + Rank A^T &= n \end{aligned}$$

and as $Rank A = Rank A^T$, we have $null A = null A^T$.

2A) Is it possible for the symmetric group S_4 to act transitively on a set with 3 elements.
Answer: The group S_4 acts by conjugation on its 3 Sylow 2-subgroups transitively by conjugation. Thus we identify the elements $\{a, b, c\}$ in the 3 element set with the Sylow subgroups P_1, P_2, and P_3.

2B) Let $GL_n(R)$ be the group of invertible matrices with real coefficients. Let V be a vector space R^3 of column vectors with coordinates x_1, x_2, x_3 and let $GL_3(R)$ act by linear transformation on V. Let $G \subseteq GL_n(R)$ be the subgroup of matrices which preserve the subspace $x_3 = 0$. Prove that there exists a normal subgroup $H \subseteq G$ such that G/H is isomorphic to $GL_2(R)$. Describe H explicitly.
Answer: Take

$$G = \left\{ \begin{pmatrix} a & b & c \\ d & e & f \\ 0 & 0 & i \end{pmatrix} : ae \neq bd \text{ and } i \neq 0 \right\}$$

Any matrix of this form clearly fixes $x_3 = 0$ and is in $GL_n(\mathbb{R})$. Through a simple computation we can show closure, inverse, and identity. Now we want a normal subgroup H such that $G/H \to GL_2(\mathbb{R})$. Take

$$H = \left\{ \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & f \\ 0 & 0 & i \end{pmatrix} : i \neq 0 \right\}$$

Do the computations to check for subgroup and normalcy and you are done.

3A) (a) Give an example of an ideal in a commutative ring with is prime but not maximal.
(b) Prove that if $f : R \to S$ is a homomorphism of commutative rings and $I \subseteq S$ is a prime ideal, then $f^{-1}(I)$ is prime. Give an example where I is maximal but $f^{-1}(I)$ is not maximal.

Answer: (a) Take $\mathbb{Q}[x,y]$ has $\langle x\rangle$ is prime as $\mathbb{Q}[x,y]/\langle x\rangle \cong \mathbb{Q}[y]$ is an integral domain. But as $\langle x\rangle \subset \langle x,y\rangle \subset \mathbb{Q}[x,y]$ and $\mathbb{Q}[x,y]/\langle x\rangle \cong \mathbb{Q}[y]$ is not a field and so $\langle x\rangle$ is not maximal.

(b) By the correspondence theorem we know that $f^{-1}(I)$ is a subring of R. Let $r_1 \in R$ and $r_2 \in f^{-1}(I)$, then $f(r_1r_2) = f(r_1)f(r_2)$. As $f(r_2) \in I$ and I is an ideal so $r_1r_2 \in f^{-1}(I)$. Let $r_1r_2 \in f^{-1}(I)$, then $f(r_1r_2) = f(r_1)f(r_2)$ so $f(r_1) \in I$ or $f(r_2) \in I$ which implies $r_1 \in f^{-1}(I)$ or $r_2 \in f^{-1}(I)$. So $f^{-1}(I)$ is a prime ideal.

Example: $f : p\mathbb{Z} \to \mathbb{Z}$ and $f^{-1}(p\mathbb{Z}) = p\mathbb{Z}$ where f is the injection map.

3B) Does there exist a ring R with 10 elements such that if $a,b \in R$, $a \neq 0$, $b \neq 0$ then $ab \neq 0$.
Answer: No there does not exist such a ring. Assume that such a ring does exist. Then R would be an integral domain except for the commutativity. So let $0 \neq a \in R$. Then for all $n \in \mathbb{N}$ we have $a^n \neq 0$. So as R is finite there must exist $m > n$ such that $a^m = a^n$. Therefore $a^{m-n} = 1$ and so a^{m-n-1} is an inverse for a. Therefore it is a division ring. Any finite division ring is a field so we have a field with 10 elements. This is a contradiction as the only finite fields have order p^n.

4A) Let $E \subseteq F \subseteq L$ be three fields. Prove or give a counterexample to the following:
(a) L is Galois over E implies L is Galois over F and F is Galois over E.
(b) L is Galois over F and F is Galois over E implies L is Galois over E.
Answer: (a) False take $\mathbb{Q}\left(2^{1/4}\right) = F$ and $L = \mathbb{Q}\left(2^{1/4}, i\right)$ and $E = \mathbb{Q}$.

(b)Take $\mathbb{Q} \subseteq \mathbb{Q}\left(2^{1/2}\right) \subseteq \mathbb{Q}\left(2^{1/4}\right)$. Each is a degree 2 extension and thus normal and it is separable as it is over a characteristic 0 field. However $\mathbb{Q}\left(2^{1/4}\right)$ is not normal as $x^4 - 2$, its minimal polynomial is not normal.

4B) Let f be a quintic polynomial with coefficients in $\mathbb{Q}$ such that the splitting field K of f has a Galois group isomorphic to the dihedral group D_5. Prove that there exists a unique quadratic field $\mathbb{Q}\left(\sqrt{d}\right) \subseteq K$.
Answer: $D_5 = \left\langle \sigma, \tau : \sigma^5 = \tau^2 = 1,\ \sigma\tau = \tau\sigma^{-1}\right\rangle$. Then $|\langle\sigma\rangle| = 5$ and so $|D_5/\langle\sigma\rangle| = 2$. $L \geq \mathbb{Q}$, $[L:\mathbb{Q}] = 2$ and implies that $L = \mathbb{Q}\left(a + b\sqrt{d}\right) = \mathbb{Q}\left(\sqrt{d}\right)$. This is unique because $\langle\sigma\rangle$ is the unique subgroup of index 2 or use the Sylow theorems and we have 1 or 6 subgroups of order 5 and so it must have one and so it must be normal.

5A) Let R be a noetherian commutative ring.
(a) Suppose $\varphi : M_1 \to M_2$ is a surjective homomorphism of R-modules. Prove that for any module N, the homomorphism $Hom_R(M_2, N) \to Hom_R(M_1, N)$ induced by composition with φ is injective.
(b) Suppoes $\varphi : N_1 \to N_2$ is a surjective homomorphism of R-modules. Prove that for any free R-module F, the homomorphism $Hom_R(F, N_1) \to Hom_R(F, N_2)$ induced by composition with φ is surjective.
Answer: (a) We want to show that $Hom_R(M_2, N) \xrightarrow{\bar{\varphi}} Hom_R(M_1, N)$ is injective:
Claim: $f \to f \circ \varphi$ has kernel of $\bar{\varphi}$ is trivial.
Proof: Take $f \in Hom_R(M_2, N)$ such that $\bar{\varphi}(f) = f \circ \varphi = 0$, i.e. for all $x \in M$ $f \circ \varphi(x) = 0$. Take any $y \in M_2$ since φ is surjective there is an $x \in M$ such that $\varphi(x) = y$.
$f \circ \varphi(x) = 0 \implies f(y) = 0$ for all $y \in M_2 \implies f$ is the zero map.

(b) Let $\{\alpha_i\}$ be a basis of F. We want to show that $Hom_R(F, N_1) \xrightarrow{\bar{\varphi}} Hom_R(F, N_2)$, $f \to \varphi f$. Take any $g \in Hom_R(F, N_2)$. We want to show there is an $f \in Hom_R(F, N)$ such that $\varphi f = g$. Say $g(\alpha_i) = a_i$. Since φ is surjective we can pick a $b_i \in N_1$ such that $\varphi(b_i) = a_i$. We want $\varphi f(\alpha_i) = a_i$. So define $f(\alpha_i) = b_i$. So $\varphi f(\alpha_i) = \varphi(b_i) = b_i$. Then by our construction, $\varphi f(\alpha_i) = a_i = g(\alpha_i)$. We check that it is a module homomorphism.

We know $g(\alpha_i + \alpha_j) = g(\alpha_i) + g(\alpha_j) = a_i + a_j$.
$\varphi \circ f(\alpha_i + \alpha_j) = g(\alpha_i + \alpha_j) = a_i + a_j$.
$g(r\alpha_i) = rg(\alpha_i) = r(\varphi f(\alpha_i)) = \varphi(rf(\alpha_i)) = ra_i$.

5B) Let R be the polynomial ring $C[T]$ and let M be the vector space of column vectors C^3. Make M into an R-module by letting C act by scalar multiplication and letting T act by the matrix $\begin{bmatrix} 2 & 7 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1+i \end{bmatrix} = A$. Write down a direct sum of cyclic R-modules which is isomorphic to M.

Answer: We first would like to do an SNF calculation for $A - xI$.

$$\begin{pmatrix} 2-x & 7 & 0 \\ 0 & 2-x & 0 \\ 0 & 0 & 1+i-x \end{pmatrix} \rightarrow \begin{pmatrix} 7 & 2-x & 0 \\ 2-x & 0 & 0 \\ 0 & 0 & 1+i-x \end{pmatrix} \rightarrow$$

$$\begin{pmatrix} 1 & 2-x & 0 \\ 2/7-x/7 & 0 & 0 \\ 0 & 0 & 1+i-x \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2-x & 0 \\ 2/7-x/7 & 0 & 0 \\ 0 & 0 & 1+i-x \end{pmatrix} \rightarrow$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 2/7-x/7 & -4/7+4/7x-x^2/7 & 0 \\ 0 & 0 & 1+i-x \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & -4/7+4/7x-x^2/7 & 0 \\ 0 & 0 & 1+i-x \end{pmatrix} \rightarrow$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -4/7+4/7x-x^2/7 & 0 \\ 0 & 0 & 1+i-x \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & x^2-4x+4 & 0 \\ 0 & 0 & 1+i-x \end{pmatrix} \rightarrow$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & (x-2)^2 & 0 \\ 0 & 0 & 1+i-x \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+i-x & 0 \\ 0 & 0 & (x-2)^2 \end{pmatrix} \rightarrow$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+i-x & 0 \\ 0 & 1+i-x & (x-2)^2 \end{pmatrix} \rightarrow \cdots \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1+i-x\,(x-2)^2 \end{pmatrix}$$

Thus we know that $M \cong \mathbb{C}[x] / \left\langle (x-2)^2 (1+i-x) \right\rangle \cong \mathbb{C}[x] / (x-2)^2 \oplus \mathbb{C}[x] / (1+i-x)$.