August 1995
Algebra Qualifying Exams

1A) Let $M = \begin{bmatrix} -12 & 18 \\ -8 & 12 \end{bmatrix}$. Find all eigenvalues of M and M^{100}.

1B) Let V be a finite dimensional vector space over a field F and let $(\ , \ ) : V \times V \to F$ be a bilinear form. Prove that

$$\dim_F \{v \in V : (v, w) = 0 \text{ for all } w \in V\} = \dim_F \{w \in V : (v, w) = 0 \text{ for all } v \in V\}.$$

2A) Is it possible for the symmetric group S_4 to act transitively on a set with 3 elements.

2B) Let $GL_n(R)$ be the group of invertible matrices with real coefficients. Let V be a vector space R^3 of column vectors with coordinates x_1, x_2, x_3 and let $GL_3(R)$ act by linear transformation on V. Let $G \subseteq GL_n(R)$ be the subgroup of matrices which preserve the subspace $x_3 = 0$. Prove that there exists a normal subgroup $H \subseteq G$ such that G/H is isomorphic to $GL_2(R)$. Describe H explicitly.

3A) (a) Give an example of an ideal in a commutative ring with is prime but not maximal.

(b) Provethat if $f : R \to S$ is a homomorphism of commutative rings and $I \subseteq S$ is a prime ideal, then $f^{-1}(I)$ is prime. Give an example where I is maximal but $f^{-1}(I)$ is not maximal.

3B) Does there exist a ring R with 10 elements such that if $a, b \in R$, $a \neq 0$, $b \neq 0$ then $ab \neq 0$.

4A) Let $E \subseteq F \subseteq L$ be three fields. Prove or give a counterexample to the following:

(a) L is Galois over E implies L is Galois over F and F is Galois over E.

(b) L is Galois over F and F is Galois over E implies L is Galois over E.

4B) Let f be a quintic polynomial with coefficients in $\mathbb{Q}$ such that the splitting field $K\ of\ f$ has a Galois group isomorphic to the dihedral group D_5. Prove that there exists a unique quadratic field $\mathbb{Q}\left(\sqrt{d}\right) \subseteq K$.

5A) Let R be a noetherian commutative ring.

(a) Suppose $\varphi : M_1 \to M_2$ is a surjective homomorphism of R-modules. Prove that for any module N, the homomorphism $Hom_R(M_2, N) \to Hom_R(M_1, N)$ induced by composition with φ is injective.

(b) Suppoes $\varphi : N_1 \to N_2$ is a surjective homomorphism of R-modules. Prove that for any free R-module F, the homomorphism $Hom_R(F, N_1) \to Hom_R(F, N_2)$ induced by composition with φ is surjective.

5B) Let R be the polynomial ring $C[T]$ and let M be the vector space of column vectors C^3. Make M into an R-module by letting C act by scalar multiplication and letting T act by the matrix $\begin{bmatrix} 2 & 7 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1+i \end{bmatrix} = A$. Write down a direct sum of cyclic R-modules which is isomorphic to M.