# August 1994
# Algebra Qualifying Exam

1A) Suppose that $A$ is an $n \times n$ complex matrix and $A^k = I$ for some $k \in \mathbb{Z}^+$. Prove that $A$ can be diagonalized.

1B) True or false; for each give either a brief reason or a counterexample.
(a) If a matrix $A$ is both Hermitian and unitary then $A = \pm I$.
(b) If $V$ is a finite dimensional vector space and $T : V \to V$ is a linear transformation then $V = \operatorname{Im}(T) \oplus \ker(T)$.
(c) Eigenvalues of orthogonal matrices are real numbers.

2A) How many essentially distinct ways can $A_4$ act transitively on a set with 3 elements?

2B) Suppose $G$ is a group, $H \leq G$ and $x^2 \in H$ for all $x \in G$. Show that $H \lhd G$ and $G/H$ is abelian.

3A) Show that every nonzero prime ideal in the ring $\mathbb{Z}[i]$ of Gaussian integers is maximal.

3B) Suppose that $R$ is a noncommutative semisimple ring and that $|R| = 81$. Describe the center of $R$ as completely as possible.

4A) If $f(x) = x^6 + x^4 - 3x^2 - 3 \in \mathbb{Q}[x]$, find a splitting field $K \subseteq \mathbb{C}$ for $f(x)$, and determine the Galois group of $f(x)$.

4B) Suppose that $f(x) \in \mathbb{Q}[x]$, $g(x) = f(x^2)$, $K \subseteq \mathbb{C}$ is a splitting field for $g(x)$ and $[K : \mathbb{Q}]$ is odd. Show that $f(x)$ and $g(x)$ have the same Galois group.

5A) If $A = \langle a, b : 45a = 63b = 105(a+b) = 0 \rangle$, then describe $A$ as a direct sum of cyclic groups and determine $|A|$.

5B) Give a proof or a counterexample.
(a) If $R$ is a PID and $M$ is a finitely generated torsion free $R$-module, then $M$ is free.
(b) If $R$ is an ID and $M$ is a finitely generated torsion free $R$-module, then $M$ is free.