August 1993
Algebra Qualifying Exam

1A) Determine all A that have distinct eigenvalues and $A^2 = 3A - 2I$.

1B) If $A = \begin{bmatrix} 6 & -2 \\ -2 & 3 \end{bmatrix}$, find an orthogonal matrix P such that $P^{-1}AP$ is diagonal.

2A) Let $\alpha = \sqrt{3 + 2\sqrt{3}}$. Find the minimal polynomial for α, the Galois closure of $\mathbb{Q}(\alpha)$, and the Galois group of the Galois closure over $\mathbb{Q}$.

2B) Let $f(x) = x^3 - 2$. Find the Galois groups over
(a) $\mathbb{Q}$
(b) $\mathbb{F}_7$
(c) $\mathbb{F}_9$

3A) Show that a finite group G generated by a and b of both order 2 is dihedral of order $2m$ for some $m \in \mathbb{Z}^+$.

3B) How many groups are there of order 63?

4A) $R = \mathbb{C}[x, y]$.
(a) Find a maximal ideal that does not contain xy.
(b) Find a prime ideal that is not maximal that does not contain xy.

4B) If A is a finitely generated $\mathbb{Z}$-module, describe $R \otimes_{\mathbb{Z}} A$ as completely as possible.

5A) (a) If R is a commutative ring, show that the set of nilpotent elements of R is an ideal in R.
(b) Prove or disprove: If R is an arbitrary ring, then the set of nilpotent elements is an ideal.

5B) Show that an ID with 1 satisfying the DCC (descending chain condition) for ideals must be a field.