August 1992
Algebra Qualifying Exam

1) Consider the system

$$\begin{array}{ccccccc} u & + & v & + & 2w & = & 0 \\ 2u & + & 3v & - & w & = & 5 \\ 3u & + & 4v & + & w & = & c \end{array}$$

For what values of c is this solvable?
Answer: By Gaussian elimination we see that $c = 5$ is the only solution.

2) Find a real matrix T, which is not diagonalizable over the reals and for which $T^7 = I$.
Answer: We know that $T^7 - I = 0$ so we have a polynomial $x^7 - 1 = 0$ and so for any $n \times n$ matrix the only possible minimal polynomials over $\mathbb{R}$ are $x - 1$ and $x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$. So if we take the RCF of $x^6 + ... + 1$ then we will have a matrix that is not diagonalizable (as the minimal polynomial does not split into distinct linear factors) but that $A^7 = 1$.

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

3) Show that a group of order 48 must have a normal subgroup of order a power of 2.
Answer: $|G| = 48 = 2^4 \cdot 3$. So there are either 1 or 3 Sylow 2-subgroups of order 16. If there is one it must be normal and we are done.If not assume that there are 3 of them say A, B, and C. The order of the subset AB is $|AB| = \dfrac{|A|\,|B|}{|A \cap B|}$. Thus $16 \cdot 16/\,|A \cap B| \leq 48 = |G|$. The only possibilities are 16 and 8. But 16 is not a possibility as we have that $A \neq B$. Thus $|A \cap B| = 8$. As $[A : A \cap B] = [B : A \cap B] = 2$ we have that $A \cap B \lhd A$ and $A \cap B \lhd B$. So $AB \leq N_G(A \cap B)$. But $|AB| = 64$ and as $N_G(A \cap B) \leq G$ it must have order 48 and thus $N_G(A \cap B) = G$ and so $A \cap B \lhd G$ and has order $8 = 2^3$.

4) Let $k$be a finite field with 7 elements. Let $f(x) = x^3 - 3$ and let α be a root of $f(x)$. Finally let $l = k(\alpha)$. Factor $f(x)$ into irreducible polynomials in $l[x]$.
Answer: Without loss of generality we can take k to be $\mathbb{Z}_7$. As $x^3 - 1 = (x + 6)(x + 3)(x + 5) \in \mathbb{Z}_7[x]$ we know that if α is a root of $x^3 - 3$ then the other roots will be $\alpha \cdot \omega$ and $\alpha \cdot \omega^2$. These are the roots $1, 2$, and 4 and so $f(x) = (x - \alpha)(x - 2\alpha)(x - 4\alpha)$.

5) Write down a principal ideal in $\mathbb{C}[x, y]$ which is not maximal. Write an ideal in $\mathbb{C}[x, y]$ which is not a principal ideal.
Answer: $\langle x \rangle$ is not a maximal ideal as $\mathbb{C}[x, y] / \langle x \rangle \cong \mathbb{C}[y]$ which is not a field. However we know that it is a prime ideal as $\mathbb{C}[y]$ is an integral domain. An ideal in $\mathbb{C}[x, y]$ which is not principal is $\langle x, y \rangle$.

6) Let G be the group of real 2×2 matrices of determinant 1, and let H be the subgroup of diagonal matrices. Find the normalizer of H in G, $N_G(H)$.
Answer:

(a) We need all matrices P such that $\det(P) = 1$ and $P^{-1}hP = h$ where $h \in H$.

$$\begin{aligned} & \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix} \begin{bmatrix} b_{22} & -b_{12} \\ -b_{21} & b_{11} \end{bmatrix} \\ = & \begin{bmatrix} b_{11}a_{11} & b_{12}a_{22} \\ b_{21}a_{11} & b_{22}a_{22} \end{bmatrix} \begin{bmatrix} b_{22} & -b_{12} \\ -b_{21} & b_{11} \end{bmatrix} \\ = & \begin{bmatrix} b_{11}a_{11}b_{22} - b_{12}a_{22}b_{21} & -b_{11}a_{11}b_{12} + b_{12}a_{22}b_{11} \\ b_{21}a_{11}b_{22} - b_{22}a_{22}b_{21} & -b_{21}a_{11}b_{12} + b_{22}a_{22}b_{11} \end{bmatrix} \end{aligned}$$

This means that

$$\begin{aligned} b_{11}a_{11}b_{22} - b_{12}a_{22}b_{21} &= a'_{11} \\ -b_{11}a_{11}b_{12} + b_{12}a_{22}b_{11} &= 0 \\ b_{21}a_{11}b_{22} - b_{22}a_{22}b_{21} &= 0 \\ -b_{21}a_{11}b_{12} + b_{22}a_{22}b_{11} &= a'_{22} \end{aligned}$$

for some a'_{11}, $a'_{22} \in \mathbb{R}$.
This implies that

$$\begin{aligned} b_{11}a_{11}b_{22} - a_{11} - b_{12}a_{22}b_{21} &= a' \\ b_{11}b_{12}\left(-a_{11} + a_{22}\right) &= 0 \text{ for all } a_{11}, a_{22} \\ b_{21}b_{22}\left(-a_{11} + a_{22}\right) &= 0 \text{ for all } a_{11}, a_{22} \end{aligned}$$

As $\mathbb{R}$ is an integral domain we know that either $b_{11} = 0$ or $b_{12} = 0$ and either $b_{21} = 0$ or $b_{22} = 0$. But we also know that $b_{11}b_{22} - b_{21}b_{12} = 1$. Thus if $b_{11} = 0$ then both b_{12} and $b_{21} \neq 1$, and thus we have matrices that look like

$$P_1 = \begin{bmatrix} a_{11} & 0 \\ 0 & a_{11}^{-1} \end{bmatrix} \text{ and } P_2 = \begin{bmatrix} 0 & a_{12} \\ -a_{12}^{-1} & 0 \end{bmatrix}$$

Thus the normalizer $N_G(H) = \langle P_1, P_2 \rangle$, the subgroup generated by the matrices P_1 and P_2.

(b) Now we need to compute coset representatives. Take a coset

7) Give an example of two non-trivial modules $M \neq \{0\}$ and $N \neq \{0\}$ over a ring R such that $M \otimes_R N = \{0\}$.
Answer: Take $M = \mathbb{Q}$, $R = \mathbb{Q}$ and $N = \mathbb{Z}_6$. We have that $\mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{Z}_6 = 0$ as

$$\frac{x}{y} \otimes z = \frac{x}{6y} \otimes 6z = \frac{x}{6y} \otimes 0 = 0$$

As another example consider $\mathbb{Z}_9 \otimes_{\mathbb{Z}} \mathbb{Z}_2$. As the gcd $(2, 9) = 1$ we know that there are $a, b \in \mathbb{Z}$ such that $a \cdot 9 + b \cdot 2 = 1$. Thus

$$\begin{aligned} x \otimes y &= (2a + 9b)\, x \otimes y = 2ax \otimes y + x \otimes 9by \\ &= 0 \otimes y + x \otimes 0 = 0 + 0 = 0 \end{aligned}$$

8) Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be an exact sequence of abelian groups. Prove: If B has torsion elements then either A or C has torsion elements.
Answer: Assume that B has torsion elements. Thus there is a $b \neq 0 \in B$ such that for some $z \neq 0 \in \mathbb{Z}$ we have that $zb = 0$. As we have an exact sequence we know that f is 1-1 and g is onto. Assume that there are no torsion elements in A. Thus for all $a \in A$ we have that $f(a) \neq 0 \in B$. Now as f is 1-1 then we know that $A \cong \operatorname{Im}(f) \leq B$. As A is torsion free thus $\operatorname{Im}(f)$ is Torsion free. Thus we know that $b \notin \operatorname{Im}(f) = \ker(g)$. Thus $g(b) \neq 0 \in C$. Say $g(b) = c \neq 0$. But $g(0) = g(zb) = zg(b) = 0$ and thus $c = g(b)$ is a Torsion element in C.

9) Suppose that T is a linear transformation on $\mathbb{C}^n$ with $T^3 = 1$. Show that T must be diagonalizable.
Answer: We have done most of these so far with the argument using that the minimal polynomial $m(x)$ for T must divide and as $x^3 - 1$ has distinct factors over $\mathbb{C}$ then so must $m(x)$ and a matrix is diagonalizable if and only if its minimal polynomial has distinct linear factors. Another way of seeing this is that T must have a JCF J as we are over $\mathbb{C}$. Thus there is an invertible matrix A such that $AJA^{-1} = T$. Then we have that $\left(AJA^{-1}\right)^3 = T^3 = I$ and thus $AJ^3A^{-1} = I$ implies that $J^3 = 1$. So the Jordan form must have only entries on the main diagonal and is thus diagonal.