August 1992
Algebra Qualifying Exam

1) Consider the system

$$\begin{array}{ccccccc} u & + & v & + & 2w & = & 0 \\ 2u & + & 3v & - & w & = & 5 \\ 3u & + & 4v & + & w & = & c \end{array}$$

For what values of c is this solvable?

2) Find a real matrix T, which is not diagonalizable over the reals and for which $T^7 = I$.

3) Show that a group of order 48 must have a normal subgroup of order a power of 2.

4) Let $k$be a finite field with 7 elements. Let $f(x) = x^3 - 3$ and let α be a root of $f(x)$. Finally let $l = k(\alpha)$. Factor $f(x)$ into irreducible polynomials in $l[x]$.

5) Write down a principal ideal in $\mathbb{C}[x, y]$ which is not maximal. Write an ideal in $\mathbb{C}[x, y]$ which is not a principal ideal.

6) Let G be the group of real 2×2 matrices of determinant 1, and let H be the subgroup of diagonal matrices.
(a) Find the normalizer of H in G, $N_G(H)$.
(b) Find the representatives for the cosets in $N_G(H)$.

7) Give an example of two non-trivial modules $M \neq \{0\}$ and $N \neq \{0\}$ over a ring R such that $M \otimes_R N = \{0\}$.

8) Let $0 \to A \to B \to C \to 0$ be an exact sequence of abelian groups. Prove: If B has torsion elements then either A or C has torsion elements.

9) Suppose that T is a linear transformation on $\mathbb{C}^n$ with $T^3 = 1$. Show that T must be diagonalizable.