Worksheet on l'Hôpital's rule

MATH 124 · Calculus I · Section 26 · Fall 2008 Name ______________________

This worksheet is designed to walk you through some problems involving l'Hôpital's rule. First I give you some general guidelines for such problems. Then, I give you some problems with substeps outlined for you. (On an exam, of course, you won't have such outlined substeps. This is practice.)

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Guidelines for using l'Hôpital's rule

Limits existing at a point

- We distinguish between the value of a function **at** a point and the limiting value of a function **near** a point.
- For example, the value of $\sin(x)$ **at** $x = \pi/2$ is $\sin(\pi/2) = 1$. We've all known about evaluating a function at a point since junior high school.
- A new concept introduced in this course is the concept of the *limit*. What does
$$\lim_{x\to\pi/2} \sin(x)$$
mean? It means that *if* we evaluate $\sin(x)$ for values of x closer and closer to $\pi/2$ (**near** $\pi/2$, from either side), and *if* those values get closer and closer to some number, the limit exists and is that number. For example:

x	$\pi/2+0.5$	$\pi/2+0.1$	$\pi/2+0.01$	$\pi/2-0.5$	$\pi/2-0.1$	$\pi/2-0.01$
$\sin(x)$	0.877583	0.995004	0.999950	0.877583	0.995004	0.999950

Since the $\sin(x)$ output values get closer and closer to 1 as we use input x values closer and closer to $\pi/2$, we say that
$$\lim_{x\to\pi/2} \sin(x) = 1.$$
- Instead of using a data table, we could also look at the graph of $\sin(x)$ to reach the same conclusion.
- The value of $\sin(x)$ **at** $\pi/2$ is separate from the limiting values of $\sin(x)$ **near** $\pi/2$. Nowhere in the above table did we compute $\sin(\pi/2)$; we only computed nearby values.
- If a function is continuous at a point (as $\sin(x)$ is), then the value of the function **at** that point is the same as the limiting value of the function **near** that point. Most functions we deal with in freshman math courses (polynomials, sine and cosine, exponential functions, etc.) *are* continuous, and so you might well wonder why we bother.
- Another way to look at this is: if you have a function which is continuous at an input point a, then
$$\lim_{x\to a} f(x) = f(a).$$
(In fact, this is the formal definition of continuity.) For example,
$$\lim_{x\to\pi/2} \sin(x) = \sin(\pi/2).$$
- For continuous functions, we can quickly find the value of the function **at** the point rather than tediously figuring out the values of the function **near** the point, since the answer will be the same.

- For functions with discontinuities, limits become useful.
- Limits are also useful when the value of a function **at** a point is not defined, but values **near** the point are. We'll see some examples soon.

Limits not existing at a point

- For another example, let's look at
$$\lim_{x\to 0} \frac{1}{x}.$$
Trying the same thing out, we get

x	0.5	0.1	0.01	-0.5	-0.1	-0.01
$1/x$	2	10	100	-2	-10	-100

Here, the output $1/x$ values don't get closer and closer to any particular number as we use input x values closer and closer to 0. So, we say
$$\lim_{x\to 0} \frac{1}{x}$$
does not exist.
- Instead of using a data table, we could also look at the graph of $1/x$ to come to the same conclusion. (Try it out on your graphing calculator.)

Limits existing at infinity

- Limits at infinity are very similar. Instead of using input x values closer and closer to some number (0 in both of the previous examples), we use input x values which are more and more positive for
$$\lim_{x\to\infty} f(x)$$
or input x values which are more and more negative for
$$\lim_{x\to-\infty} f(x).$$
- Let's see what
$$\lim_{x\to\infty} 1/x$$
might be.

x	-1	-10	-100	-200	-500	-1000
$1/x$	-1.0	-0.1	-0.01	-0.005	-0.002	-0.001

Based on this evidence, we say that
$$\lim_{x\to\infty} 1/x = 0.$$
- On the graph as well, as x goes to the left, $1/x$ gets closer and closer to zero.

Limits not existing at infinity

- What about
$$\lim_{x\to-\infty} \sin(x)?$$

x	-1	-10	-100	-200	-500	-1000
$\sin(x)$	-0.841471	0.544021	0.506366	0.873297	0.467772	-0.826880

As we use input x values more and more negative, the $\sin(x)$ values don't settle down on anything.
- Instead of using a data table, we could also look at the graph of $\sin(x)$ to come to the same conclusion: the sine function's output values keep oscillating between -1 and 1.

What does 0/0 mean?

- In order to know when to use l'Hôpital's rule, you need to know when it applies and when it doesn't. First, l'Hôpital's rule will apply for evaluating limits of functions. Second, it will only work for functions which are written as fractions, or which can be written as fractions. For example,
$$\lim_{x\to\infty} \frac{x^2+1}{2x^2-3x+4}, \qquad \text{or} \qquad \lim_{x\to 0} x\ln(x) \quad \text{which can be written as} \quad \lim_{x\to 0} \frac{\ln(x)}{1/x}.$$
- The 0/0 case occurs when you try to evaluate the numerator and denominator separately and you get a zero on the top and the bottom. For example
$$\lim_{x\to 0} \frac{\sin(x)}{x}.$$
Since the numerator and denominator are continuous functions, we can try to evaluate
$$\lim_{x\to 0} \frac{\sin(x)}{x} = \frac{\lim_{x\to 0} \sin(x)}{\lim_{x\to 0} x} = \frac{\sin(0)}{0} = \frac{0}{0}.$$
Remember that 0/0 means "you don't know yet."

What does ∞/∞ mean?

- Let's try to evaluate
$$\lim_{x\to 0} \frac{\ln(x)}{1/x}.$$
(Technically, this is only a right-handed limit.) As before, we can try to move the limit through the fraction bar. We get
$$\lim_{x\to 0} \frac{\ln(x)}{1/x} = \frac{\lim_{x\to 0} \ln(x)}{\lim_{x\to 0} 1/x}.$$
Now, $\ln(x)$ and $1/x$ aren't continuous at 0. And evaluating the numerator and denominator at $x = 0$ we get undefined results, as your calculator will quickly tell you. But if you try taking $\ln(x)$ and $1/x$ for input x values closer and closer to zero, you'll get output values which get bigger and bigger for the numerator and denominator.

x	.5	.2	.1	.01	.001	.0001	.00001
$\ln(x)$	-0.693147	-1.609438	-2.302585	-4.605170	-6.907755	-9.210340	-11.512925
$1/x$	2	5	10	100	1000	10000	100000

For this reason, we say
$$\lim_{x\to 0} \frac{\ln(x)}{1/x}$$
is an ∞/∞ case.
- Clearly we can't evaluate this function **at** zero. But we can evaluate the function **near** zero and see what (if anything) it settles down to.

x	.5	.2	.1	.01	.001	.0001	.00001
$\frac{\ln(x)}{1/x}$	-0.346574	-0.321888	-0.230259	-0.046052	-0.006908	-0.000921	-0.000115

- Based on this evidence, or the graph of $x\ln(x)$ (try it out on your graphing calculator), you might have already concluded that
$$\lim_{x\to 0} x\ln(x) = \lim_{x\to 0} \frac{\ln(x)}{1/x} = 0.$$

Why bother?

- In the example we just saw, you might have guessed the above limit from the graph of $x \ln x$. You would be correct, and you might wonder why we need l'Hôpital's rule to tell you something you can already read off the graph. I would mostly agree with you.
- One added value of l'Hôpital's rule is that it can tell you exact values of limits. If a numerical experiment tells you a certain limit looks like, say, 0.693147, that might not seem like a special number. But if you use l'Hôpital's rule and find that the limit is exactly $\ln(2)$, then you know something you didn't know before.
- Also, l'Hôpital's rule helps you when parameters are involved — e.g. it will tell you what
$$\lim_{x\to 0} \frac{\sin(ax)}{bx}$$
is. This is something your calculator can't do. (At least, my calculator can't.)

l'Hopital's rule

- Here's the rule:
  - **If** you have a function which can be written as a fraction, and
  - **if** you are trying to find the limit of that function as the input approaches a point or infinity, i.e. $\lim_{x\to a} \frac{f(x)}{g(x)}$ or $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ or $\lim_{x\to-\infty} \frac{f(x)}{g(x)}$, and
  - **if** when you try to evaluate the limit you get a $0/0$ or ∞/∞ case,
  - **then** you can write
  $$\lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)}.$$
  Here, I write just lim for $\lim_{x\to a}$ or $\lim_{x\to\pm\infty}$.
- (This seems rather special-purpose, and it is. You don't need to use this rule all the time, and even when you do need it, it doesn't always apply when you'd hoped. However, it comes up often enough in math, science, and engineering that we include it in a first-semester calculus syllabus.)
- You may or may not be done. **If** the above list of if's applies to $\lim(f'(x)/g'(x))$, **then** you have the opportunity to do it all over again:
$$\lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)} = \lim \frac{f''(x)}{g''(x)}.$$
You can keep re-applying the rule for as long as the above list of if's applies.
- Note that you really do get to differentiate the top and bottom separately. Don't use the quotient rule.
- Example:
$$\lim_{x\to 0} \frac{\sin(x)}{x}$$
is a $0/0$ case. We can write
$$\lim_{x\to 0} \frac{\sin(x)}{x} = \lim_{x\to 0} \frac{\cos(x)}{1.} = \frac{\lim_{x\to 0}\cos(x)}{\lim_{x\to 0} 1} = \frac{\cos(0)}{1} = \frac{1}{1} = 1.$$
- Example:
$$\lim_{x\to 0} \frac{\ln(x)}{1/x}$$
is an ∞/∞ case. We can write
$$\lim_{x\to 0} \frac{\ln(x)}{1/x} = \lim_{x\to 0} \frac{1/x}{-1/x^2} = \lim_{x\to 0}(-x) = 0.$$

- Example:
$$\lim_{x\to\infty} \frac{x^2+1}{2x^2-3x+4}$$
is an ∞/∞ case. We can write
$$\lim_{x\to\infty} \frac{x^2+1}{2x^2-3x+4} = \lim_{x\to\infty} \frac{2x}{4x-3}.$$
This is also an ∞/∞ case, so we can differentiate top and bottom again:
$$\lim_{x\to\infty} \frac{x^2+1}{2x^2-3x+4} = \lim_{x\to\infty} \frac{2x}{4x-3} = \lim_{x\to\infty} \frac{2}{4} = \frac{1}{2}.$$

When l'Hôpital's rule does and doesn't apply

- On your calculator, graph $\sin(x)/x$. From the graph, it's clear that
$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1 \qquad \text{and} \qquad \lim_{x\to\infty} \frac{\sin(x)}{x} = 0.$$
- Try to evaluate $\sin(x)/x$ at 0. You get a 0/0 case and so l'Hôpital's rule applies. Then
$$\lim_{x\to 0} \frac{\sin(x)}{x} \lim_{x\to 0} \frac{\cos(x)}{1}.$$
We can try to evaluate this. We can move the limit through the fraction bar to get
$$\lim_{x\to 0} \frac{\cos(x)}{1} = \frac{\lim_{x\to 0}\cos(x)}{\lim_{x\to 0} 1}.$$
And since $\cos(x)$ and 1 are continuous functions, we can replace the limits with function values:
$$\lim_{x\to 0} \frac{\cos(x)}{1} = \frac{\lim_{x\to 0}\cos(x)}{\lim_{x\to 0} 1} = \frac{\cos(0)}{1} = 1.$$
We're done, and l'Hôpital's rule worked. It told us what we already guessed from the graph.
- Flush with our success, we might try to find
$$\lim_{x\to\infty} \frac{\sin(x)}{x}.$$
Differentiating top and bottom, we get
$$\lim_{x\to\infty} \frac{\cos(x)}{1} = \lim_{x\to\infty} \cos(x).$$
As we try bigger and bigger x values, $\cos(x)$ keeps oscillating between -1 and 1, never settling down on anything. What went wrong? The problem is that
$$\lim_{x\to\infty} \frac{\sin(x)}{x}$$
was neither a 0/0 case nor an ∞/∞ case. The hypotheses (the list of if's) of l'Hôpital's rule are not satisfied, so it could not help us. We need to use the graph, or other methods of reasoning, to evaluate the limit.

Problem 1. Let

$$f(z) = \frac{3^z}{z^3}.$$

Using pencil and paper (not using your calculator), fill out the following table:

z	-2	-1	0	1	2
$f(z)$					

Using your calculator, fill out the following table (using for example 2nd CALC VALUE, or 2nd TBLSET and 2nd TABLE, in your calculator):

z	-2	-1	0	1	2
$f(z)$					

Graph the function on your calculator using the standard window. Are values of $f(z)$, for z near 0, approaching any value? If so, what? If not, why not?

Evaluate the numerator of $f(z)$ at $z = 0$. What is it?

Evaluate the denominator of $f(z)$ at $z = 0$. What is it?

Does l'Hôpital's rule apply here to find $\lim_{z\to 0} f(z)$?

Go ahead and use l'Hôpital's rule to find $\lim_{z\to 0} f(z)$.

Does your answer for the previous part match what you see on the graph?

Problem 2. Let

$$g(x) = \frac{x^2}{\sinh(x)}.$$

Using your calculator, fill out the following table:

x	-1	-0.1	-0.01	0	0.01	0.1	1
$g(x)$							

Graph the function on your calculator using the standard window. Are values of $g(x)$, for x near 0, approaching any value? If so, what? If not, why not?

Evaluate the numerator of $g(x)$ at $x = 0$. What is it?

Evaluate the denominator of $g(x)$ at $x = 0$. What is it?

Does l'Hôpital's rule apply here for $\lim_{x\to 0} g(x)$?

Use l'Hôpital's rule to find $\lim_{x\to 0} g(x)$.

Does your answer for the previous part match what you see on the graph?

Problem 3. Let

$$g(x) = \frac{x^2}{\sinh(x)}.$$

Using your calculator, fill out the following table (using for example 2nd CALC VALUE, or 2nd TBLSET and 2nd TABLE, in your calculator):

x	1	10	100	1000
$g(x)$				

Graph the function on your calculator using the standard window. Are values of $g(x)$, for x approaching infinity, approaching any value? If so, what? If not, why not?

Evaluate the numerator of $g(x)$ at $x = \infty$. What is it? Notice that there's no ∞ button on your calculator …you need to figure out (how is up to you) what the numerator is doing as $x \to \infty$.

Evaluate the denominator of $g(x)$ at $x = \infty$. What is it?

Does l'Hôpital's rule apply here for $\lim_{x\to\infty} g(x)$?

Go ahead and use l'Hôpital's rule to find $\lim_{x\to 0} g(x)$.

Does your answer for the previous part match what you see on the graph?

Problem 4. Find

$$\lim_{x\to\infty} \frac{x^3}{\sinh(x)}$$

using l'Hôpital's rule.

Problem 5. Find

$$\lim_{x\to\infty} \frac{x^4}{\sinh(x)}$$

using l'Hôpital's rule.

Problem 6. From looking at a graph, pencil-and-paper experiments, or what have you, what do you think

$$\lim_{x \to 0} \frac{1}{x}$$

is?

Does l'Hôpital's rule help you to confirm that guess?

Problem 7. From looking at a graph, pencil-and-paper experiments, or what have you, what do you think

$$\lim_{x \to -\infty} \frac{1}{x}$$

is?

Does l'Hôpital's rule help you to confirm that guess?

Problem 8. From looking at a graph, what do you think

$$\lim_{x \to 0} x \ln(x)$$

is?

Does l'Hôpital's rule help you to confirm that guess? Write it as

$$\lim_{x \to 0} \frac{\ln(x)}{1/x}$$

and work it out.

Problem 9. If a is a constant and b is a non-zero constant, what is

$$\lim_{x \to 0} \frac{\sin(ax)}{bx}?$$