Exam #2 solutions · Thursday, October 9, 2008

MATH 124 · Calculus I · Section 26 · Fall 2008

Note: Some of my solutions are wordy, for the sake of explanation. All I expect of you is computation, except for questions which specifically ask for verbal responses.

John Kerl (`kerl at math dot arizona dot edu`).

Problem 1. Let $f(x) = x^{2x}$. Numerically approximate $f'(1)$ using difference quotients. Use at least three successively smaller values of h.

Solution: We numerically approximate derivatives by computing

$$\frac{f(x+h) - f(x)}{h}$$

for the specified x and smaller and smaller h's. Here, this is

$$\frac{(x+h)^{2(x+h)} - x^{2x}}{h} = \frac{(1+h)^{2(1+h)} - 1^2}{h} = \frac{(1+h)^{2(1+h)} - 1}{h}$$

$$\begin{aligned}
h = 0.1: &\quad \frac{1.1^{2.2} - 1}{0.1} \approx 2.332863 \\
h = 0.01: &\quad \frac{1.01^{2.02} - 1}{0.01} \approx 2.030303 \\
h = 0.001: &\quad \frac{1.001^{2.002} - 1}{0.001} \approx 2.003003.
\end{aligned}$$

This is beginning to look like 2.

Problem 2. The period T (in seconds) of a pendulum depends on the length ℓ (in centimeters) of the pendulum, as follows:

$$T = 2\pi\sqrt{\frac{\ell}{g}}$$

where g is the acceleration due to gravity (a positive constant).

Part (a). What are the units of $dT/d\ell$?

Solution: Units of a derivative are the original function's output units over the original function's input units. Here, this is seconds per centimeter. Alternatively, the Leibniz-style $dT/d\ell$ notation suggests putting T's units over ℓ's units, i.e. seconds per centimeter.

Part (b). Find the formula for $dT/d\ell$.

Solution:

$$\begin{aligned}
\frac{dT}{d\ell} &= \frac{d}{d\ell}\left(2\pi\sqrt{\frac{\ell}{g}}\right) \\
&= 2\pi\frac{d}{d\ell}\left(\left(\frac{\ell}{g}\right)^{1/2}\right) && \text{(multiplicative constants pull through)} \\
&= 2\pi\frac{1}{2}\left(\frac{\ell}{g}\right)^{-1/2}\frac{d}{d\ell}\left(\frac{\ell}{g}\right) && \text{(chain and power rules)} \\
&= 2\pi\frac{1}{2}\left(\frac{\ell}{g}\right)^{-1/2}\frac{1}{g} \\
&= \frac{\pi}{g}\left(\frac{\ell}{g}\right)^{-1/2} && \text{(some simplification to clean up)} \\
&= \frac{\pi}{g}\sqrt{\frac{g}{\ell}} \\
&= \frac{\pi}{\sqrt{g\ell}}.
\end{aligned}$$

Alternatively, notice that $\sqrt{\ell/g} = \sqrt{\ell}/\sqrt{g}$. Then

$$\begin{aligned}
\frac{dT}{d\ell} &= \frac{d}{d\ell}\left(\frac{2\pi}{\sqrt{g}}\sqrt{\ell}\right) \\
&= \frac{2\pi}{\sqrt{g}}\frac{d}{d\ell}\left(\sqrt{\ell}\right) \\
&= \frac{2\pi}{\sqrt{g}}\frac{d}{d\ell}\left((\ell)^{1/2}\right) \\
&= \frac{2\pi}{\sqrt{g}}\frac{1}{2}(\ell)^{-1/2} \\
&= \frac{\pi}{\sqrt{g}\sqrt{\ell}} \\
&= \frac{\pi}{\sqrt{g\ell}}.
\end{aligned}$$

Problem 3. The horizontal displacement H (in millimeters) of a weighted spring from its equilibrium position, at time t milliseconds from the start of an experiment, is

$$H(t) = 2.3 + e^{-2t}\cos(t).$$

Part (a). What are the units of $H''(t)$?
Solution: Units of the second derivative are the original function's output units over the original functions input units squared. (As before, the Leibniz notation d^2H/dt^2 is a helpful device to remember this fact.) Here, this is millimeters per millisecond squared: mm/ms^2.
Part (b). Find $H''(t)$.

Solution:

$$\begin{aligned}
H(t) &= 2.3 + e^{-2t}\cos(t). \\
H'(t) &= \frac{d}{dt}(e^{-2t}\cos(t)) \\
&= \frac{d}{dt}(e^{-2t})\cos(t) + e^{-2t}\frac{d}{dt}(\cos(t)) && \text{(product rule)} \\
&= -2e^{-2t}\cos(t) - e^{-2t}\sin(t) && \text{(chain and trig rules)} \\
H''(t) &= -2\frac{d}{dt}(e^{-2t}\cos(t)) - \frac{d}{dt}(e^{-2t}\sin(t)) \\
&= -2\frac{d}{dt}(e^{-2t})\cos(t) - 2e^{-2t}\frac{d}{dt}\cos(t) - \frac{d}{dt}(e^{-2t})\sin(t) - e^{-2t}\frac{d}{dt}(\sin(t)) \\
&= 4e^{-2t}\cos(t) + 2e^{-2t}\sin(t) + 2e^{-2t}\sin(t) - e^{-2t}\cos(t) \\
&= 3e^{-2t}\cos(t) + 4e^{-2t}\sin(t) \\
&= e^{-2t}(3\cos(t) + 4\sin(t).
\end{aligned}$$

Several of you found $H'(t)$ as above, then factored out the e^{-2t} to obtain

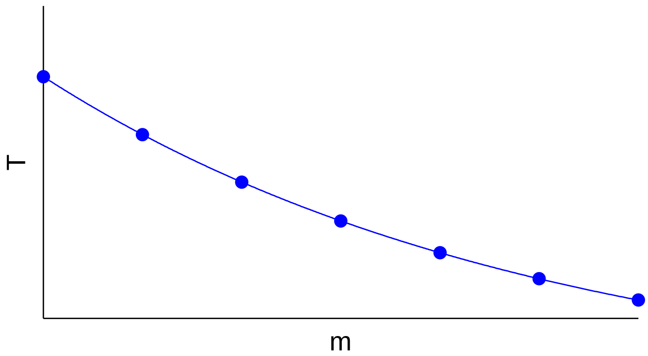
$$H'(t) = e^{-2t}(-2\cos(t) - \sin(t)).$$

Then we can use the product rule to get

$$\begin{aligned}
H''(t) &= -2e^{-2t}(-2\cos(t) - \sin(t)) + e^{-2t}(2\sin(t) - \cos(t)) \\
&= e^{-2t}(4\cos(t) + 2\sin(t) + 2\sin(t) - \cos(t)) \\
&= e^{-2t}(3\cos(t) + 4\sin(t)).
\end{aligned}$$

You get the same answer either way, but as it happens, the latter works out a little nicer.
Problem 4. The reaction time T (in minutes) of a chemical experiment is a function of the amount m (in grams) of catalyst added to a solution. That is, $T = f(m)$.

Part (a). Describe the significance of $f(200)$, including units in your answer.
Solution: f takes grams to minutes. So, this is number of minutes the reaction takes to complete when 200 grams of catalyst are used.
Part (b). Describe the significance of $f^{-1}(6)$, including units in your answer.
Solution: f^{-1} takes minutes to grams. So, this is the number of grams of catalyst needed to make the reaction complete in 6 minutes.
Part (c). Suppose you discover the following: As you repeat the experiment, with one more gram of catalyst each time, you find that the reaction time decreases. However, each additional gram has less and less effect. What is sign of $f''(m)$? Please justify your answer. (Sketch a graph if you like. If you do so, please label your axes and explain verbally what you are doing.)
Solution: Here's a sketch which is compatible with the information presented: the reaction time T decreases as m increases, but decreases by less and less each time:



From the sketch it is clear that the T vs. m curve is concave up, i.e. $f''(m)$ is positive.
A non-graphical way to think of this is that $f'(m)$ is negative, but less and less so as m increases. That is, $f''(m)$ is positive.

Problem 5. Let $f(x) = \frac{1}{x^2+1}$.

Part (a). Find $f'(x)$.
Solution:

$$\begin{aligned}\frac{d}{dx}\left(\frac{1}{x^2+1}\right) &= \frac{(x^2+1)(0)-(1)(2x)}{(x^2+1)^2} \\ &= \frac{-2x}{(x^2+1)^2}.\end{aligned}$$

Part (b). Find an equation for the tangent line to $f(x)$ at $x = 2$.
Solution: Using point-slope form, the tangent line has equation

$$y = m(x-a)+b.$$

We have $a = 2$, $b = f(2)$, and $m = f'(2)$. The latter two are

$$\begin{aligned}f(2) &= \frac{1}{2^2+1} = \frac{1}{5} \\ f'(2) &= \frac{-2(2)}{(2^2+1)^2} = \frac{-4}{25}.\end{aligned}$$

Then

$$\begin{aligned}y &= \frac{-4}{25}(x-2)+\frac{1}{5} \\ &= \frac{-4x}{25}+\frac{8}{25}+\frac{1}{5} \\ &= \frac{-4x}{25}+\frac{13}{25} \\ &= -0.16x+0.52 \qquad \text{(if you prefer).}\end{aligned}$$

Problem 6. Some values of $V(t)$ and $H(t)$ and their first derivatives are given by the following table.

t	$V(t)$	$H(t)$	$V'(t)$	$H'(t)$
6	8	-7	0.5	-0.2

Part (a). Let $E(t) = V(t) + H(t)$. Find $E'(6)$.
Solution: Using the sum rule, we have $E'(t) = V'(t) + H'(t)$ and thus $E'(6) = V'(6) + H'(6)$. Reading the values off the table, this is $0.5 - 0.2 = 0.3$.

Part (b). Let $E(t) = V(t)H(t)$. Find $E'(6)$.
Solution: Using the product rule, we have $E'(t) = V'(t)H(t)+V(t)H'(t)$ and thus $E'(6) = V'(6)H(6)+V(6)H'(6)$. Reading the values off the table, this is $0.5(-7)+6(-0.2) = -3.5-1.2 = -4.7$.

Part (c). Let $E(t) = \frac{V(t)}{H(t)}$. Find $E'(6)$.
Solution: Using the product rule, we have

$$E'(t) = \frac{V'(t)H(t) - V(t)H'(t)}{H(t)^2}$$

and thus

$$E'(6) = \frac{V'(6)H(6) - V(6)H'(6)}{H(6)^2}.$$

Reading the values off the table, this is

$$\frac{(0.5)(-7) - (8)(-0.2)}{(-7)^2} = \frac{-3.5 + 1.6}{49} = \frac{-1.9}{49}.$$

Problem 7. Let $g(x) = xe^{-ax}$. Assume a is a positive constant. Over what interval(s) is $g(x)$ increasing? (You will need to solve an inequality.)
Solution: The function $g'(x)$ is increasing when its derivative $g'(x)$ is positive. We need to solve the inequality $g'(x) > 0$. First, we need to find $g'(x)$:

$$\begin{aligned} g'(x) &= \frac{d}{dx}(x)e^{-ax} + x\frac{d}{dx}(e^{-ax}) && \text{(product rule)} \\ &= e^{-ax} - axe^{-ax} && \text{(chain rule)} \\ &= (1 - ax)e^{-ax}. \end{aligned}$$

Then the inequality:

$$\begin{aligned} (1 - ax)e^{-ax} &> 0 \\ (1 - ax) &> 0 && (e^{\text{anything}} \text{ is positive}) \\ 1 &> ax \\ x &< 1/a && (a \text{ is positive}). \end{aligned}$$

Problem 8. Let $C(\theta) = \frac{\cos(\theta) - 1}{\theta}$.

Part (a). Does $C'(1)$ exist? If so, find its value exactly. If not, explain why not.
Solution: A function is differentiable at a point ("innocent until proven guilty") unless one of the following happens:

- *The original function isn't defined at that point*: No, since we can take cosine minus 1 of anything, and we can divide by θ as long as it's non-zero — which is the case since $\theta = 1$.
- *Discontinuity at that point*: No, as may be checked from the graph.
- *A corner at that point*: No, as may be checked from the graph.
- *A vertical tangent line at that point*: No, as may be checked from the graph.

We can use the quotient rule to find the derivative:

$$\begin{aligned} C'(\theta) &= \frac{\theta(-\sin(\theta)) - (\cos\theta - 1)(1)}{\theta}^2 \\ &= \frac{-\theta\sin(\theta) - \cos\theta + 1}{\theta}^2 \\ C'(1) &= -\sin(1) - \cos(1) + 1. \end{aligned}$$

Part (b). Does $C'(0)$ exist? If so, find its value exactly. If not, explain why not.
Solution: $C(0)$ is not defined (division by zero) so the derivative cannot be defined there either.