Appendix A

Bose-gas derivation of random permutations

In this sketch, we motivate the otherwise ab-initio construction of the model of random spatial permutations in chapter 2. More details may be found in [BU07, U07]. As above, we write $\mathbf{X} = (\mathbf{x}_1, \ldots, \mathbf{x}_N)$ for $\mathbf{x}_1, \ldots, \mathbf{x}_N$ in a d-dimensional cube Λ of width L. The Hamiltonian for N pair-interacting particles is

$$\mathbf{H}(\mathbf{X}) = -\sum_{i=1}^{N} \nabla_i^2 + \sum_{1 \le i,j \le N} U(\mathbf{x}_i - \mathbf{x}_j). \tag{A.0.1}$$

The U considered here is either identically zero (for the non-interacting case), or a hard-core potential with radius a, i.e. $U(\mathbf{x}_i - \mathbf{x}_j)$ is infinite for $|\mathbf{x}_i - \mathbf{x}_j| \le a$ and zero for $|\mathbf{x}_i - \mathbf{x}_j| > a$. (This is an approximation to the true pair potential between helium atoms. See figure A.1 [Ceperley].) The hard-core radius a is also known as the scattering length.

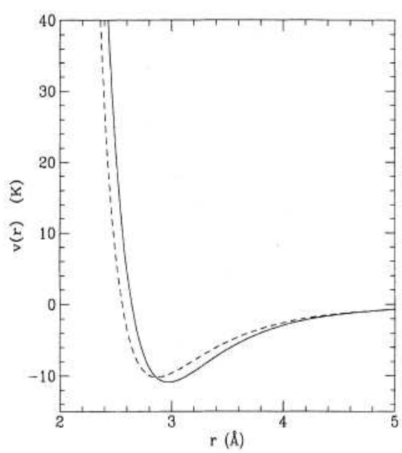

Figure A.1. Pair potential between helium atoms (Ceperley, 1995).

The partition function for N distinguishable particles[1] is $\mathrm{Tr}(e^{-\beta\mathbf{H}})$. Symmetrizing the partitition function, since our particles are bosons, the trace is

$$\mathrm{Tr}_{L^2_{\mathrm{sym}}}(e^{-\beta\mathbf{H}}) = \mathrm{Tr}_{L^2}\left(P_+ e^{-\beta\mathbf{H}}\right) = \mathrm{Tr}_{L^2}\left(e^{-\beta\mathbf{H}} P_+\right)$$

[1]For a particle Hamiltonian, the $\beta = 1/T$ factor is in the expected place. This is in contrast to the permutation expression in chapter 2, where the β factor is, surprisingly, reciprocated. As discussed in [BU07, U07], the reciprocated β is correct for the permutation Hamiltonian.

where

$$P_+ f(\mathbf{x}_1, \ldots, \mathbf{x}_N) := \frac{1}{N!} \sum_{\pi \in \mathcal{S}_n} M_\pi f(\mathbf{x}_1, \ldots, \mathbf{x}_N)$$

and

$$M_\pi(f\mathbf{x}_1, \ldots, \mathbf{x}_N) := f(\mathbf{x}_{\pi(1)}, \ldots, \mathbf{x}_{\pi(N)}).$$

That is,

$$\mathrm{Tr}_{L^2_{\mathrm{sym}}}(e^{-\beta \mathbf{H}}) = \frac{1}{N!} \sum_{\pi \in \mathcal{S}_N} \mathrm{Tr}_{L^2}\left(e^{-\beta \mathbf{H}} M_\pi\right).$$

(The operator $e^{-\beta \mathbf{H}}$ is bounded and compact, but this fact is not needed.)

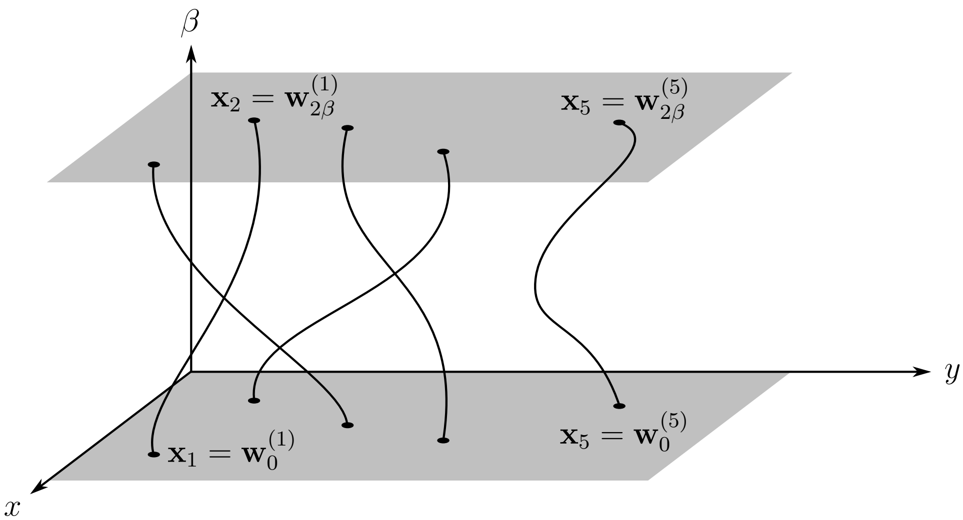

FIGURE A.2. Feynman-Kac representation of a gas of 5 bosons. The horizontal plane represents the d spatial dimensions, and the vertical axis is the imaginary time dimension. The picture shows five particles and two cycles, of respective length 4 and 1.

The following steps are involved in developing a bosonic Feynman-Kac formula. The first three steps closely parallel the steps used to construct the familiar single-particle Feynman-Kac formula. (1) Interpret $e^{-\beta \mathbf{H}} M_\pi$ as an expectation over Brownian motions. (2) Write $e^{-\beta \mathbf{H}} M_\pi$ as an integral operator, and find the kernel. (3) Compute $\mathrm{Tr}(e^{-\beta \mathbf{H}} M_\pi)$ in terms of Brownian bridges. (4) Sum over $\pi \in \mathcal{S}_N$ to obtain $Z = \mathrm{Tr}_{L^2_{\mathrm{sym}}}(e^{-\beta \mathbf{H}})$. Importantly, one expresses Z as sum over permutations π of $e^{-H_P(\mathbf{X},\pi)}$, where this new H_P will be viewed as a Hamiltonian for permuations π. At this point, the permutation Hamiltonian is found inside $e^{-H_P(\mathbf{X},\pi)}$; one lacks an expression for its logarithm. (5) Decouple the non-interacting terms from the interacting terms in the permutation Hamiltonian, so that one may write $e^{-H_P^{(0)}(\mathbf{X},\pi) - H_P^{(1)}(\mathbf{X},\pi)}$.

The bosonic Feynman-Kac formula now contains terms for two-jump interactions, three-jump interactions, and so on. (6) A cluster expansion allows one to drop all but two-jump interactions. The cluster expansion furthermore allows one to take the logarithm of $e^{-H_P(\mathbf{X},\pi)}$, with an explicit expression for $H_P(\mathbf{X},\pi)$. (7) One recognizes the random-cycle model from equation (2.1.1) of chapter 2, with an explicit two-jump interaction V. Specifically, given one permutation jump from $\mathbf{x}_i$ to $\mathbf{x}_{\pi(i)}$ and another permutation jump from $\mathbf{x}_j$ to $\mathbf{x}_{\pi(j)}$, the two-jump interaction $V(\mathbf{x}_i, \mathbf{x}_{\pi(i)}, \mathbf{x}_j, \mathbf{x}_{\pi(j)})$ involves the probability that two Brownian bridges, running in time 2β from $\mathbf{x}_i$ to $\mathbf{x}_{\pi(i)}$ and $\mathbf{x}_j$ to $\mathbf{x}_{\pi(j)}$, respectively, pass within distance $2a$ from one another.