Notes for representation theory

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Abstract

This is a crib sheet for representation theory. It is under construction.

Contents

1 Basic formulas and orthogonality relations

$\chi(g) = \mathrm{Tr}(U(g))$

$$\begin{aligned}
\langle \chi_i, \chi_j \rangle &= \frac{1}{o(G)} \sum_{g \in G} \overline{\chi_i(g)} \chi_j(g) = \delta_{ij} \\
&= \frac{1}{o(G)} \sum_{C_k} \#(C_k) \overline{\chi_i(C_k)} \chi_j(C_k) = \delta_{ij}
\end{aligned}$$

$$\frac{1}{o(G)} \sum_{g \in G} \overline{D_{ij}^{(\alpha)}(g)} D_{k\ell}^{(\beta)}(g) = \frac{1}{d_\alpha} \delta_{\alpha\beta} \delta_{ik} \delta_{j\ell}$$

$$\{\sqrt{d_\alpha} D_{ij}^{(\alpha)}(g)\}$$

is an o.n. basis for $\mathbb{C}G$, with inner product

$$\langle f, g \rangle = \frac{1}{o(G)} \sum_{x \in G} \overline{f(x)} g(x).$$

$\dim(\mathbb{C}G) = o(G)$

$\#(\hat{(G)}) =$ number of cjg classses of G.

$\{\chi^{(\alpha)}\}$ form an o.n. basis for $Z(\mathbb{C}G)$.

characters are class functions

Inner product on characters:

$$\langle \chi_i, \chi_j \rangle = \frac{1}{o(G)} \sum_{g \in G} \overline{\chi_i(g)} \chi_j(g) = \delta_{ij}.$$

Orthogonality:

$$\sum_{\chi} \overline{\chi(C_i)} \chi(C_j) = \frac{\delta_{ij} o(G)}{\#C_j};$$

$$\sum_{C} \chi^{(\alpha)}(C) \chi^{(\beta)}(C) \frac{\#C}{o(G)} = \delta_{\alpha\beta}.$$

2 Character table for $\mathcal{S}_3$

3 Induced representations

To write or not to write?

4 Induced characters

Type up the nice homework here.

Frobenius character formula ....

5 The little-group method

Following the method of section V.1 of [**Simon**], we compute the character table for $\mathcal{V}_4 \rtimes \mathcal{S}_3$.

In general, the method of the constructive proof of Simon's proposition V.1.1. applies to $G = N \rtimes_\alpha H$ where $N \lhd G$, N is abelian, and $H \to \mathrm{Aut}(N)$. Here we have $N = \mathcal{V}_4 = \{e, a, b, c\}$, and α has $H = \mathcal{S}_3$ permuting $\{a, b, c\}$.

Recall that the dual of $\mathcal{V}_4$, i.e. $\hat{\mathcal{V}}_4$, is

	e	a	b	c
χ_0	1	1	1	1
χ_1	1	1	-1	-1
χ_2	1	-1	1	-1
χ_3	1	-1	-1	1

Then we form $\beta : H \to \mathrm{Aut}(\hat{N})$ by

$$(\beta_h \chi)(n) = \chi(\alpha_h^{-1} n).$$

Here, we have

	e	a	b	c
$\beta_1\chi_0$	1	1	1	1
$\beta_1\chi_1$	1	1	-1	-1
$\beta_1\chi_2$	1	-1	1	-1
$\beta_1\chi_3$	1	-1	-1	1

	e	a	b	c
$\beta_{(12)}\chi_0$	1	1	1	1
$\beta_{(12)}\chi_1$	1	-1	1	-1
$\beta_{(12)}\chi_2$	1	1	-1	-1
$\beta_{(12)}\chi_3$	1	-1	-1	1

	e	a	b	c
$\beta_{(123)}\chi_0$	1	1	1	1
$\beta_{(123)}\chi_1$	1	-1	1	-1
$\beta_{(123)}\chi_2$	1	-1	-1	1
$\beta_{(123)}\chi_3$	1	1	-1	-1

and likewise for the remaining three elements of $\mathcal{S}_3$, namely, β_h acts by permuting the second through fourth columns of the character table. Thus we read off the action of β on $\hat{\mathcal{V}}_4$:

	χ_0	χ_1	χ_2	χ_3
β_1	χ_0	χ_1	χ_2	χ_3
$\beta_{(12)}$	χ_0	χ_2	χ_1	χ_3
$\beta_{(13)}$	χ_0	χ_3	χ_2	χ_1
$\beta_{(23)}$	χ_0	χ_1	χ_3	χ_2
$\beta_{(123)}$	χ_0	χ_2	χ_3	χ_1
$\beta_{(132)}$	χ_0	χ_3	χ_1	χ_2

Thus the orbits of $\hat{\mathcal{V}}4$ under β are

$$\mathcal{O}_0 = \{\chi_0\}, \quad \mathcal{O}_1 = \{\chi_1, \chi_2, \chi_3\}.$$

For these two orbits I choose representatives χ_0 and χ_1, respectively.

Next we compute the isotropy subgroups H_i for χ_i. From the table above, we see that all elements of $\mathcal{S}_3$ fix the trivial character of $\mathcal{V}_4$; only the identity permutation and (23) fix χ_1. That is,

$$H_0 = \mathcal{S}_3, \qquad H_1 = \langle (23) \rangle.$$

The construction of irreducible *representations* U for $N \rtimes H$ requires knowledge of irreducible *representations* L for H_i. However, we wish only to compute irreducible *characters* χ_U for $N \rtimes H$ and it happens that we only need irreducible *characters* χ_L for H_i. Thus we skip most of step 1 of Simon's proof, which involves the construction of U's.

Recall the character tables for $H_0 = \mathcal{S}_3$ and $H_1 = \langle(23)\rangle \cong \mathbb{Z}_2$:

	1	(12)	(123)
$\chi_{L\mathcal{S}_3,1}$	1	1	1
$\chi_{L\mathcal{S}_3,2}$	1	-1	1
$\chi_{L\mathcal{S}_3,3}$	2	0	-1

	1	(23)
$\chi_{L\langle(23)\rangle,1}$	1	1
$\chi_{L\langle(23)\rangle,2}$	1	-1

Following Simon's notation, we compute cosets x of H/H_i:

$$\begin{array}{ll} H_0: & x = \mathcal{S}_3; \\ H_1: & x = \{1,(23)\},\{(12),(123)\},\{(13),(132)\}. \end{array}$$

I select the following coset representatives:

$$\begin{array}{ll} H_0: & h_x = \mathcal{S}_3; \\ H_1: & h_x = (23),(12),(13). \end{array}$$

We should have $\chi_x = (\beta_{h_x})\chi_i \in \mathcal{O}_i$ being distinct and exhausting all of $\mathcal{O}_i$. In fact, we compute

$$\begin{array}{ll} H_0: & \chi_x = \chi_0; \\ H_1: & \chi_x = \chi_1,\chi_2,\chi_3 \end{array}$$

which furnishes a sanity check.

For each choice of $\mathcal{O}_i$ and χ_L we will have an irreducible character χ_U of $\mathcal{V}_4 \rtimes \mathcal{S}_3$. For $\mathcal{O}_0$ there are three χ_L's; for $\mathcal{O}_1$ there are two χ_L's. The dimensions should turn out to be $\dim(L)\cdot\#(H/H_i)$. These dimensions are then $1\cdot 1 = 1$, $1\cdot 1 = 1$, and $2\cdot 1 = 2$ for $\mathcal{O}_0$, and $1\cdot 3 = 3$ and $1\cdot 3 = 3$ for $\mathcal{O}_1$. This will furnish another sanity check below.

Next we compute $H_x = h_x H_i h_x^{-1}$. Recalling the values of h_x above, we have

$$\begin{array}{rcl} H_0: 1\mathcal{S}_3 1 &=& \mathcal{S}_3, \\ H_1: (23)\langle(23)\rangle(23) &=& \langle(23)\rangle, \\ H_1: (12)\langle(23)\rangle(12) &=& \langle(13)\rangle, \\ H_1: (13)\langle(23)\rangle(13) &=& \langle(12)\rangle. \end{array}$$

Simon's formula for χ_U requires the indicator function $Ch_x : \mathcal{S}_3 \to \{0,1\}$:

$$Ch_x(h) = \begin{cases} 1, & h \in H_x; \\ 0, & h \notin H_x. \end{cases}$$

Here, we have

	1	(12)	(13)	(23)	(123)	(132)
H_1	1	1	1	1	1	1
$H_{(23)}$	1	0	0	1	0	0
$H_{(12)}$	1	0	1	0	0	0
$H_{(13)}$	1	1	0	0	0	0

Having tabulated these data, we are almost ready to apply Simon's formula V.1.7 for χ_U's:

$$\chi_U(n,h) \;=\; \sum_{x\in\mathcal{O}_i} \chi_x(n)\; Ch_x(h)\; \chi_L(h_x^{-1} h h_x).$$

Since characters are class functions, we do not need to compute these sums for all 24 elements of $\mathcal{V}_4 \rtimes \mathcal{S}_3$ (i.e. 4 values of n and 6 values of h). Recall from appendix B that we have the following conjugacy classes for $\mathcal{V}_4 \rtimes \mathcal{S}_3$:

$$\begin{aligned}
C_1 &= \{e1\},\\
C_2 &= \{a1, b1, c1\},\\
C_3 &= \{e12, e13, e23, a23, b13, c12\},\\
C_4 &= \{a12, b12, a13, c13, b23, c23\},\\
C_5 &= \{e123, a123, b123, c123, e132, a132, b132, c132\}.
\end{aligned}$$

I select the first element of each conjugacy class as representatives. Now we may apply the formula. Having already carefully tabulated the needed data, we may simply (and with low risk of computational error) read off the desired values.

(1) χ_{U1} comes from $\mathcal{O}_0$ and L the trivial representation on $\mathcal{S}_3$; $h_x = 1$.

n	h		$\chi_x(n)$		$Ch_x(h)$		$\chi_L(h_x^{-1}hh_x)$	
$\chi_{U1}(e,$	$1)$	$=$	$\chi_0(e)$	$\cdot$	1	$\cdot$	1	$=1$
$\chi_{U1}(a,$	$1)$	$=$	$\chi_0(a)$	$\cdot$	1	$\cdot$	1	$=1$
$\chi_{U1}(e,$	$12)$	$=$	$\chi_0(e)$	$\cdot$	1	$\cdot$	1	$=1$
$\chi_{U1}(a,$	$12)$	$=$	$\chi_0(a)$	$\cdot$	1	$\cdot$	1	$=1$
$\chi_{U1}(e,$	$123)$	$=$	$\chi_0(e)$	$\cdot$	1	$\cdot$	1	$=1$

(2) χ_{U2} comes from $\mathcal{O}_0$ and L the parity representation on $\mathcal{S}_3$; $h_x = 1$.

n	h		$\chi_x(n)$		$Ch_x(h)$		$\chi_L(h_x^{-1}hh_x)$	
$\chi_{U2}(e,$	$1)$	$=$	$\chi_0(e)$	$\cdot$	1	$\cdot$	1	$=1$
$\chi_{U2}(a,$	$1)$	$=$	$\chi_0(a)$	$\cdot$	1	$\cdot$	1	$=1$
$\chi_{U2}(e,$	$12)$	$=$	$\chi_0(e)$	$\cdot$	1	$\cdot$	-1	$=-1$
$\chi_{U2}(a,$	$12)$	$=$	$\chi_0(a)$	$\cdot$	1	$\cdot$	-1	$=-1$
$\chi_{U2}(e,$	$123)$	$=$	$\chi_0(e)$	$\cdot$	1	$\cdot$	1	$=1$

(3) χ_{U3} comes from $\mathcal{O}_0$ and L the two-dimensional representation on $\mathcal{S}_3$; $h_x = 1$.

n	h		$\chi_x(n)$		$Ch_x(h)$		$\chi_L(h_x^{-1}hh_x)$	
$\chi_{U3}(e,$	$1)$	$=$	$\chi_0(e)$	$\cdot$	1	$\cdot$	2	$=2$
$\chi_{U3}(a,$	$1)$	$=$	$\chi_0(a)$	$\cdot$	1	$\cdot$	2	$=2$
$\chi_{U3}(e,$	$12)$	$=$	$\chi_0(e)$	$\cdot$	1	$\cdot$	0	$=0$
$\chi_{U3}(a,$	$12)$	$=$	$\chi_0(a)$	$\cdot$	1	$\cdot$	0	$=0$
$\chi_{U3}(e,$	$123)$	$=$	$\chi_0(e)$	$\cdot$	1	$\cdot$	1	$=-1$

(4) χ_{U4} comes from $\mathcal{O}_1$ and L the trivial representation on $\langle(23)\rangle$; $h_x = (23), (12), (13)$.

n	h		$\chi_x(n)$		$Ch_x(h)$		$\chi_L(h_x^{-1}hh_x)$	
$\chi_{U4}(e,$	$1)$	$=$	$\chi_1(e)$	$\cdot$	1	$\cdot$	1	
		$+$	$\chi_2(e)$	$\cdot$	1	$\cdot$	1	
		$+$	$\chi_3(e)$	$\cdot$	1	$\cdot$	1	$= 1 + 1 + 1 = 3$
$\chi_{U4}(a,$	$1)$	$=$	$\chi_1(a)$	$\cdot$	1	$\cdot$	1	
		$+$	$\chi_2(a)$	$\cdot$	1	$\cdot$	1	
		$+$	$\chi_3(a)$	$\cdot$	1	$\cdot$	1	$= 1 - 1 - 1 = -1$
$\chi_{U4}(e,$	$12)$	$=$	$\chi_1(e)$	$\cdot$	0			
		$+$	$\chi_2(e)$	$\cdot$	0			
		$+$	$\chi_3(e)$	$\cdot$	1	$\cdot$	$\chi_L((13)(12)(13)$	$= \chi_L(23) = 1$
$\chi_{U4}(a,$	$12)$	$=$	$\chi_1(a)$	$\cdot$	0			
		$+$	$\chi_2(a)$	$\cdot$	0			
		$+$	$\chi_3(a)$	$\cdot$	1	$\cdot$	$\chi_L((13)(12)(13)$	$= -\chi_L(23) = -1$
$\chi_{U4}(e,$	$123)$	$=$	$\chi_1(e)$	$\cdot$	0			
		$+$	$\chi_2(e)$	$\cdot$	0			
		$+$	$\chi_3(e)$	$\cdot$	0			$= 0$

(5) χ_{U5} comes from $\mathcal{O}_1$ and L the parity representation on $\langle(23)\rangle$; $h_x = (23), (12), (13)$.

n	h		$\chi_x(n)$		$Ch_x(h)$		$\chi_L(h_x^{-1}hh_x)$	
$\chi_{U5}(e,$	$1)$	$=$	$\chi_1(e)$	$\cdot$	1	$\cdot$	1	
		$+$	$\chi_2(e)$	$\cdot$	1	$\cdot$	1	
		$+$	$\chi_3(e)$	$\cdot$	1	$\cdot$	1	$= 1 + 1 + 1 = 3$
$\chi_{U5}(a,$	$1)$	$=$	$\chi_1(a)$	$\cdot$	1	$\cdot$	1	
		$+$	$\chi_2(a)$	$\cdot$	1	$\cdot$	1	
		$+$	$\chi_3(a)$	$\cdot$	1	$\cdot$	1	$= 1 - 1 - 1 = -1$
$\chi_{U5}(e,$	$12)$	$=$	$\chi_1(e)$	$\cdot$	0			
		$+$	$\chi_2(e)$	$\cdot$	0			
		$+$	$\chi_3(e)$	$\cdot$	1	$\cdot$	$\chi_L((13)(12)(13)$	$= \chi_L(23) = -1$
$\chi_{U5}(a,$	$12)$	$=$	$\chi_1(a)$	$\cdot$	0			
		$+$	$\chi_2(a)$	$\cdot$	0			
		$+$	$\chi_3(a)$	$\cdot$	1	$\cdot$	$\chi_L((13)(12)(13)$	$= -\chi_L(23) = 1$
$\chi_{U5}(e,$	$123)$	$=$	$\chi_1(e)$	$\cdot$	0			
		$+$	$\chi_2(e)$	$\cdot$	0			
		$+$	$\chi_3(e)$	$\cdot$	0			$= 0$

Collecting these results, we have the character table for $\mathcal{V}_4 \rtimes \mathcal{S}_3$:

	$e1$	$a1$	$e12$	$a12$	$e123$
χ_{U1}	1	1	1	1	1
χ_{U2}	1	1	-1	-1	1
χ_{U3}	2	2	0	0	-1
χ_{U4}	3	-1	1	-1	0
χ_{U5}	3	-1	-1	1	0

A Semidirect products

(This section is merely review material from abstract algebra.)

Definition A.1. Let G be a group with $N \lhd G$ and $K \subseteq G$. Furthermore suppose that $G = NK$ and $N \cap K = \{1\}$. If K acts on N (formally, if there is a homomorphism $K \to \mathrm{Aut}(N)$), we write

$$N \rtimes K$$

and call this the **semidirect product** of N and K. We sometimes write the action of $k \in K$ on $n \in N$ as n^k. Since $G = NK$, any $g \in G$ may be written in the form (n, k). Multiplication in G is

$$(n_1, k_1)(n_2, k_2) = (n_1 {n_2}^{k_1}, k_1 k_2).$$

Remark A.2. How does inversion work? Given (n, k) we need to find (m, j) such that $(n, k)(m, j) = (1, 1)$. This gives

$$(1, 1) = (n, k)(m, j) = (nm^k, kj)$$

which forces $j = k^{-1}$. Then, in the first slot, $nm^k = 1$ forces

$$\begin{aligned} m^k &= n^{-1} \\ m &= (n^{-1})^{k^{-1}}. \end{aligned}$$

Thus,

$$(n, k)^{-1} = ((n^{-1})^{k^{-1}}, k^{-1}).$$

Example A.3. $\triangleright$ Dihedral groups may be written as

$$\mathcal{D}_n = \mathcal{C}_n \rtimes \mathcal{C}_2 = \langle \rho \rangle \rtimes \langle \phi \rangle$$

where ρ is the rotation of order n and ϕ is the flip. The flip acts on the rotations by inverting them:

$$\rho^i \phi^j \rho^k \phi^\ell = \begin{cases} \rho^{i+k} \phi^{j+\ell}, & j \text{ even} \\ \rho^{i-k} \phi^{j+\ell}, & j \text{ odd} \end{cases}$$

which is to say (since there are only two distinct powers of ϕ)

$$\begin{aligned} \rho^i \rho^k \phi^\ell &= \rho^{i+k} \phi^\ell \\ \rho^i \phi \rho^k \phi^\ell &= \rho^{i-k} \phi^{1+\ell}. \end{aligned}$$

This is the familiar transposition rule

$$\phi \rho^k = \rho^{-k} \phi.$$

$\triangleleft$

Example A.4. $\triangleright$ The T group (the other non-abelian group of order 12, besides $\mathcal{A}_4$ and $\mathcal{D}_6$) is

$$T = \mathbb{Z}_3 \rtimes \mathbb{Z}_4$$

where $\mathbb{Z}_4$ acts on $\mathbb{Z}_3$ by inversion. That is, 1 and 3 in $\mathbb{Z}_4$ negate elements of $\mathbb{Z}_3$; 0 and 2 in $\mathbb{Z}_4$ leave elements of $\mathbb{Z}_3$ alone. $\triangleleft$

B Arithmetic in $\mathcal{V}_4 \rtimes \mathcal{S}_3$

Some of the examples in this note make use of $\mathcal{V}_4 \rtimes \mathcal{S}_3$. Here I work out arithmetic for this group.

Let
$$\mathcal{V}_4 = \{e, a, b, c\}$$
be the **Klein-four group** as usual. Recall that **symmetric group** $\mathcal{S}_3$ is
$$\mathcal{S}_3 = \{1, (12), (13), (23), (123), (132)\}.$$

Let $\mathcal{S}_3$ act on $\mathcal{V}_4$ by permuting the symbols a, b, and c. For example, $a^{(123)} = b$. Then we can write the semidirect product
$$\mathcal{V}_4 \rtimes \mathcal{S}_3.$$

It can be shown (I won't here) that this is isomorphic to $\mathcal{S}_4$.

Let $u, v \in \mathcal{V}_4$ and $\sigma, \tau \in \mathcal{S}_3$. Using definition A.1, we can write the product
$$(u, \sigma)(v, \tau) = (uv^{\sigma}, \sigma\tau).$$

For example,
$$\begin{array}{rcl} (a, (123))(b, (12)) & = & (ab^{123}), (123)(12)) \\ & = & (ac, (13)) \\ & = & (b, 13). \end{array}$$

In particular, we have a **square formula**
$$(u, \sigma)^2 = (uu^{\sigma}, \sigma^2)$$
and a **cube formula**
$$\begin{array}{rcl} (u, \sigma)^3 & = & (uu^{\sigma}, \sigma^2)(u, \sigma) \\ & = & (uu^{\sigma}u^{\sigma^2}, \sigma^3). \end{array}$$

Also from remark A.2, and recalling that all elements of $\mathcal{V}_4$ are their own inverses, we have an **inverse formula**
$$(u, \sigma)^{-1} = (u^{\sigma^{-1}}, \sigma^{-1}).$$

From this we obtain a **conjugation formula**
$$\begin{array}{rcl} (v, \tau)^{-1}(u, \sigma)(v, \tau) & = & (v^{\tau^{-1}}, \tau^{-1})(uv^{\sigma}, \sigma\tau) \\ & = & (v^{\tau^{-1}}(uv^{\sigma})^{\tau^{-1}}, \tau^{-1}\sigma\tau). \end{array}$$

xxx conjugacy classes: first compute element orders ....

References

[**Grove**] L.C. Grove. *Algebra.* Dover, 2004.

[**Simon**] B. Simon *Representations of Finite and Compact Groups.* American Mathematical Society, 1996.

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