# Riemannian geometry notes

John Kerl

February 3, 2008

**Abstract**

These are some random jottings on Riemannian geometry. Complete information may be found in Lee's *Riemannian Manifolds: An Introduction to Curvature*.

# Contents

**Contents** 2

**1 Overview** 3

**2 Notation** 4

**3 Matrix view of vectors, covectors, linear transformations, and metrics** 6

**4 Contraction, flats, and sharps** 7

**5 Formulas** 9

5.1 Notation . . . 9

5.2 Riemannian metric . . . 9

5.3 Christoffel symbols, connections, and covariant derivatives . . . 9

5.4 Section title TBD . . . 10

5.5 Section title TBD . . . 10

5.6 Curvatures . . . 11

5.7 To be filed . . . 12

**References** 13

**Index** 14

# 1 Overview

Here I want to spell out in a little more detail than [**Lee3**] what sharps, flats, contraction, and tensor inner products look like in coordinates. In particular, I want to show how things look in terms of row vectors, column vectors, and matrices whenever possible. I could work this out on scratch paper (and I have done so), but (as is my wont) it seems a shame not to typeset these useful things for legible future reference.

## 2 Notation

In this section I spell out some details of the Einstein summation convention. Let $(M, g)$ be a Riemannian $m$-manifold. We are working in the category of smooth manifolds, so when I say *map* or *section*, I mean *smooth map* and *smooth section*.

**Vector fields:** Let $X, Y$ be vector fields on $M$. These are sections of the tangent bundle $TM$. A *collection of several vectors* will be indexed by subscripts. In particular, in coordinates defined on an open subset of $M$, $TM$ is spanned by

$$\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^m}.$$

Those look like superscripts but they're in the denominator so we think of them as subscripts.

The *components of a single vector* will be indexed by superscripts. In particular, the $m$ coordinates for a point $\mathbf{q}$ of $M$, in a given coordinate chart, will be written

$$x^1, \ldots, x^m.$$

Likewise, since the $\partial/\partial x^j$'s span $TM$, each $X$ is a linear combination thereof with coefficients $X^j$:

$$X = \sum_{j=1}^{m} X^j \frac{\partial}{\partial x^j}.$$

The **Einstein summation convention** is used: If the same index appears repeated in an expression, once in a subscript and once in a superscript, then the summation is implicit. We write

$$X = X^j \frac{\partial}{\partial x^j}.$$

**Covector fields:** Let $\lambda, \mu$ be covector fields (1-forms) on $M$. These are sections of the tangent bundle $TM$. A *collection of several covectors* will be indexed by subscripts. In particular, in coordinates defined on an open subset of $M$, $T^*M$ is spanned by

$$dx^1, \ldots, dx^m.$$

The *components of a single covector* will be indexed by subscripts. Since the $dx^i$'s span $T^*M$, each $\lambda$ is a linear combination thereof with coefficients $X^i$:

$$\lambda = \sum_{i=1}^{m} \lambda_i dx^i.$$

Again, using Einstein summation, this is

$$\lambda = \lambda_i dx^i.$$

**Tensor fields:** A tensor of type $\binom{k}{\ell}$ has $k$ covariant components and $\ell$ contravariant components. A covector field has type $\binom{1}{0}$ and is covariant:

$$\lambda \in T^1(M)$$

a vector field has type $\binom{0}{1}$ and is contravariant:

$$X \in T_1(M);$$

The above index convention applies: the covariant components of a tensor are superscripts (or denominator subscripts), and the contravariant components of a tensor are subscript (or denominator superscripts).

**Metric tensor fields:** We usually call these simply *metric tensors*. The metric 2-tensor on the Riemannian manifold $(M, g)$ is $g$. It is a symmetric positive-definite doubly covariant 2-tensor; it is a bilinear map from $TM \times TM$ to $\mathbb{R}$. We write

$$g(X, Y).$$

Since $g$ is covariant, the components of $g$ are $g_{ij}$ for $1 \leq i, j \leq m$. Since the components of $X$ and $Y$ are $X^i$ and $Y^i$, respectively, and using linearity, we have

$$\begin{aligned} g(X, Y) &= g\left(X^i \frac{\partial}{\partial x^i}, Y^j \frac{\partial}{\partial x^j}\right) \\ &= X^i Y^j g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right). \end{aligned}$$

We write

$$g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right).$$

and call these the *components* of $g$. Then we have simply

$$g(X, Y) = g_{ij} X^i Y^j.$$

## 3 Matrix view of vectors, covectors, linear transformations, and metrics

**Remark 3.1.** Recall that $\lambda(X)$ is, in coordinates,

$$\lambda(X) = \lambda_i X^i.$$

Since covectors are row vectors and vectors are column vectors, we have

$$\lambda(X) = \begin{pmatrix} \lambda_1 & \dots & \lambda_m \end{pmatrix} \begin{pmatrix} X^1 \\ \vdots \\ X^m \end{pmatrix}.$$

**Remark 3.2.** Let $A$ be a linear transformation on $TM$. Recall that $A(X)$ is, in coordinates,

$$A(X) = A^i_j X^j.$$

From the matrix point of view, we have

$$A(X) = \begin{pmatrix} A^1_1 & \dots & A^1_m \\ \vdots & & \vdots \\ A^m_1 & \dots & A^m_m \end{pmatrix} \begin{pmatrix} X_1 \\ \vdots \\ X_m \end{pmatrix}.$$

That is, a matrix times a column vector is another column vector.

**Remark 3.3.** Likewise, if $\lambda$ is a row vector, then $\lambda A$ is another row vector: In coordinates,

$$\lambda A = \lambda_i A^i_j.$$

From the matrix point of view, we have

$$\lambda A = \begin{pmatrix} \lambda_1 & \dots & \lambda_m \end{pmatrix} \begin{pmatrix} A^1_1 & \dots & A^1_m \\ \vdots & & \vdots \\ A^m_1 & \dots & A^m_m \end{pmatrix}.$$

**Remark 3.4.** We can think of a linear transformation as a **mixed tensor** of type 1-1. This means it must consume a covector and a vector, as follows:

$$A(\lambda, X) = \lambda A X.$$

Note that the expression on the right is well-defined without parentheses due to the commutativity of matrix multiplication. In coordinates,

$$\lambda A X = \lambda_i A^i_j X^j.$$

From the matrix point of view, we have

$$\lambda A X = \begin{pmatrix} \lambda_1 & \dots & \lambda_m \end{pmatrix} \begin{pmatrix} A^1_1 & \dots & A^1_m \\ \vdots & & \vdots \\ A^m_1 & \dots & A^m_m \end{pmatrix} \begin{pmatrix} X_1 \\ \vdots \\ X_m \end{pmatrix}.$$

**Remark 3.5.** This is superficially similar to computing $g(X, Y)$:

$$g(X, Y) = g_{ij} X^i Y^j.$$

Now, $g$ does *not* change coordinates in the same way as a matrix $A$; $g$ is of type $\binom{2}{0}$ and $A$ is of type $\binom{1}{1}$. Nonetheless, for fixed coordinates, if we think of the two-dimensional array $g_{ij}$ as a matrix $G$, then we can think of $g(X, Y)$ as

$$g(X, Y) = X^t G Y.$$

## 4 Contraction, flats, and sharps

**Definition 4.1.** The **flat operator** or **index-lowering operator** takes vectors to covectors:

$$X \mapsto X^\flat$$

such that for all $Y$,

$$X^\flat(Y) = g(X, Y).$$

**Remark 4.2.** In coordinates, the right-hand side is

$$g(X, Y) = g_{ij} X^i Y^j$$

and the left-hand side is (since $X^\flat$ is a covector)

$$X^\flat(Y) = X^\flat_j Y^j.$$

Setting these two equal means that

$$X^\flat_j = g_{ij} X^i.$$

According to [**Lee3**], it is standard practice to write

$$X^\flat = X_j dx^j,$$

so we have simply

$$X_j = g_{ij} X^i.$$

As matrices, we have

$$X^\flat = X^t G.$$

**Definition 4.3.** The **sharp operator** or **index-raising operator** takes covectors to vectors:

$$\lambda \mapsto \lambda^\sharp$$

such that for all $Y$,

$$\lambda(Y) = g(\lambda^\sharp, Y).$$

This is constructed to be the inverse of the flat operator.

**Remark 4.4.** In coordinates, the left-hand side is

$$\lambda(Y) = \lambda_j Y^j$$

and the right-hand side is (since $\lambda^\sharp$ is a vector)

$$g(\lambda^\sharp, Y) = g_{ij} (\lambda^\sharp)^i Y^j.$$

Setting these two equal means that

$$\lambda_j = g_{ij} (\lambda^\sharp)^i.$$

Apparently, it is also standard practice to write

$$\lambda^\sharp = \lambda^i \frac{\partial}{\partial x^i}.$$

Then we write simply

$$\lambda_j = g_{ij} \lambda^i.$$

As matrices, we have

$$\lambda = (\lambda^\sharp)^t G.$$

Now, since $G$ is invertible, this is the same as

$$\lambda G^{-1} = (\lambda^\sharp)^t$$

which as a sum is

$$\lambda_i g^{ij} = \lambda^j$$

as long as we define $g_{ij}$ to be the components of $G$ and $g^{ij}$ to be the components of $G^{-1}$. **Note:** This doesn't match Lee, but I can't find my mistake (or his).

**Definition 4.5.** **trace** or **contraction** ....

# 5 Formulas

These are formulas involving metrics, covariant derivatives, Christoffel symbols, and various types of curvature.

## 5.1 Notation

Throughout let

$$\{E_i\}_{i=1}^n$$

be a local frame on an $n$-dimensional Riemannian manifold $(M, g)$. In particular,

$$\left\{\frac{\partial}{\partial x^i}\right\}_{i=1}^n$$

is a local frame with respect to a coordinate $\mathbf{x}$. For brevity, I will write this as

$$\{\partial_i\}_{i=1}^n .$$

Note that the denominator superscript $i$ has become a subscript $i$ outside of a fraction, in keeping with the Einstein convention.

The **Lie bracket** of two vector fields $X$ and $Y$ is

$$[X, Y] = XY - YX. \tag{5.1}$$

The **product rule** gives

$$X(fY) = X(f)Y + fXY. \tag{5.2}$$

Tensors are listed as type $\binom{k}{\ell}$ where the upper valence is covariant and the lower valence is contravariant. In particular, a $k$-form is a tensor of type $\binom{k}{0}$, a vector field is of type $\binom{0}{1}$, and the Riemannian metric is of type $\binom{2}{0}$.

## 5.2 Riemannian metric

The **Riemannian metric**:

$$g_{ij} = g(E_i, E_j). \tag{5.3}$$

Recall that $g_{ij}$ is symmetric in $i$ and $j$. This is an $n \times n$ array. Treated as a matrix, it has an inverse (since it is positive definite). This is written

$$g^{ij}. \tag{5.4}$$

## 5.3 Christoffel symbols, connections, and covariant derivatives

**Christoffel symbols** in terms of the metric:

$$\Gamma_{ij}^k = \frac{1}{2} g^{km} \left(\partial_i g_{jm} + \partial_j g_{im} - \partial_m g_{ij}\right). \tag{5.5}$$

Recall that $\Gamma^k_{ij}$ is symmetric in $i$ and $j$.

A **connection** $\nabla_X Y$ satisifies:

$$\nabla_{fX} Y = f \nabla_X Y \quad \text{and additivity in X} \tag{5.6}$$
$$\nabla_X (fY) = X(f) Y + f \nabla_X Y \quad \text{and additivity in Y} \tag{5.7}$$
$$\tag{5.8}$$

This defines a connection $\nabla$:

$$\nabla_{E_i} E_j = \Gamma^k_{ij} E_k \tag{5.9}$$

with extension by linearity. xxx fill in the steps:

$$\nabla_X Y = (XY^k + X^i Y^j \Gamma^k_{ij}) E_k. \tag{5.10}$$

**Compability** of covariant derivative $\nabla$ and metric $g$:

$$\partial_k g\left(\partial_i, \partial_j\right) = g\left(\nabla_{\partial_k} \partial_i, \partial_j\right) + g\left(\partial_i, \nabla_{\partial_k} \partial_j\right). \tag{5.11}$$

**Torsion** $\binom{2}{1}$-tensor:

$$\tau(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]. \tag{5.12}$$

xxx cmt on torsion-freeness of the Levi-Civita connection:

$$\nabla_X Y - \nabla_Y X = [X, Y]. \tag{5.13}$$

## 5.4 Section title TBD

**Shape operator**:

10-09 notes: II stuff ....

xxx:

$$II(X, Y) = g(X, Y) \hat{\mathbf{n}} \tag{5.14}$$

with $(\tilde{M}, \tilde{g})$ and $(M, g)$ setup.

## 5.5 Section title TBD

**Riemann measure**:

$$\left| \sqrt{\det(g)}\, dx^1 \wedge \cdots \wedge dx^n \right|. \tag{5.15}$$

**Riemann volume**:

$$\sqrt{\det(g)}\, dx^1 \wedge \cdots \wedge dx^n. \tag{5.16}$$

## 5.6 Curvatures

xxx:

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{5.17}$$

xxx (or Rm?):

$$R(X,Y,Z,W) = g(R(X,Y)Z,W) \tag{5.18}$$
$$= R_{ijk\ell} X^i Y^j Z^k W^\ell \tag{5.19}$$

Dimension 2: There is only one sectional curvature $R_{1221}$.

$$\mathrm{Ric}(X,Y) = Kg(X,Y); S = 2K. \tag{5.20}$$

10-11 notes: principal, mean, sectional curvatures

10-11 notes: **Gauss equation**

10-16 notes: Ric and $R$ on a frame.

**Bianchi identities**: Define

$$\nabla_R(X,Y,Z,W,V) = \nabla_V R(X,Y,Z,W).$$

Then

$$R(X,Y,Z,W) + R(Y,Z,X,W) + R(Z,X,Y,W) = 0 \tag{5.21}$$
$$\nabla_R(X,Y,Z,V,W) + \nabla_R(Y,V,Z,W,X) + \nabla_R(V,X,Z,W,Y) = 0. \tag{5.22}$$

xxx 3 symmetries of $R$:

$$R(X,Y,Z,W) = -R(Y,X,Z,W) \tag{5.23}$$
$$R(X,Y,Z,W) = -R(X,Y,W,Z) \tag{5.24}$$
$$R(X,Y,Z,W) = +R(W,Z,X,Y). \tag{5.25}$$

**Sectional curvature**:

$$K(X,Y) = \frac{R(X,Y,Y,X)}{g(X,X) + g(Y,Y) - g(X,Y)^2}. \tag{5.26}$$

xxx:

$$R_{ijk}{}^{\ell} = \partial_i \Gamma^\ell_{jk} - \partial_j \Gamma^\ell_{ij} + \Gamma^m_{ik}\Gamma^\ell_{jm} - \Gamma^m_{jk}\Gamma^\ell_{im}. \tag{5.27}$$

xxx:

$$R = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}. \tag{5.28}$$

**Scalar curvature** $S$ or $R$:

$$S = g^{ij} R_{ij} = g^{k\ell} g^{ij} R_{kij\ell}. \tag{5.29}$$

**Ricci flow**:

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij}. \tag{5.30}$$

### 5.7 To be filed

xxx comment on (with $\phi$ a 1-form):

$$d\phi(X,Y) = X\phi(Y) - Y\phi(X) - \phi([X,Y]). \tag{5.31}$$

xxx ??:

$$g\left(\nabla_X Y, Z\right) = \frac{1}{2}\left(Xg(Y,Z) + \ldots \text{(look up in Lee)}\right). \tag{5.32}$$

## References

[**Lee3**] J. Lee. *Riemannian Manifolds: An Introduction to Curvature*. Springer, 1997.

# Index

**B**
Bianchi identities ........ 11

**C**
Christoffel symbols ........ 9
compatibility ........ 10
connection ........ 10
contraction ........ 8
Covector fields: ........ 4

**E**
Einstein summation convention ........ 4

**F**
flat operator ........ 7

**G**
Gauss equation ........ 11

**I**
index-lowering operator ........ 7
index-raising operator ........ 7

**L**
Lie bracket ........ 9

**M**
Metric tensor fields: ........ 4
mixed tensor ........ 6

**P**
product rule ........ 9

**R**
Ricci flow ........ 11
Riemann measure ........ 10
Riemann volume ........ 10
Riemannian metric ........ 9

**S**
scalar curvature ........ 11
sectional curvature ........ 11
shape operator ........ 10
sharp operator ........ 7
summation convention ........ 4

**T**
Tensor fields: ........ 4
torsion ........ 10
trace ........ 8

**V**
Vector fields: ........ 4