Notes for quantum networking theory

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Abstract

This is a reference sheet, vocabulary sheet, and to-do list for my spring 2008 independent study course under Janek Wehr on quantum networking theory. This paper is under construction.

Contents

xxx 541 notes!!!

xxx cite 541

xxx intro-to-QM (fin dim) disclaimer — for me as learner.

xxx “states” vs. [???] — make the pre-observation and post-observation vocabulary clear and consistent.

xxx change (.,.) to bra-ket notation throughout.

1 Introduction

This is a reference sheet, vocabulary sheet, and to-do list for my spring 2008 independent study course under Janek Wehr on quantum networking theory. For notation and terminology, please see [**Ker1**] (probability) and [**Ker2**] (basic quantum mechanics).

2 Single qubits

2.1 Single-qubit state space

$$|\,0\,\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |\,1\,\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

two-level quantum system. Make the distinction between physical implementations (which vary) and the mathematical abstraction (which is clean and simple).

Bloch sphere. Include figure ...

measurements. Two uses: input preparation, and output assessment.

Need algebra of matrices to invent “really good measurements”.

2.2 Single-qubit operators

Observable: self-adjoint operator on the state space.

Measurement: the observable’s spectral projector.

Work out some examples.

2.3 Lack of time evolution for quantum computation

Time evolution is generated by the **Hamiltonian**. Here the Hamiltonian is zero: $e^{iHt} = I$. Or eigenstate of the Hamiltonian.

Cite [**Ker2**].

physics kitchen . . . type up notes.

3 Multiple qubits

3.1 Multiple-qubit states

linear combinations

binary/unary diagram — a key concept (for me anyway).

entanglement as indecomposability of tensor.

EPR pair ...

measurements of entanglement

maximally entangled

Bell state

singlet conversion

singlet conversion probability

pure states and **mixed states**. **density operator**.

emph distinction between pure state, mixed state, and entangled state.

$$\rho = \sum_j p_j \mid \phi_j \rangle\langle \phi_j \mid$$

with $sum_j p_j = 1$. Note that ρ is said to be a **trace-class operator**. [xxx def and xref. This is a trivial distinction in finite dimensions.] [xxx $p_j = \langle\psi_j \mid \psi_j\rangle = \|\psi_j\|^2$?]

Pure state: rank-one projection. Iff $\rho^2 = \rho$.

$$\langle A \rangle = \sum_j p_j \langle\phi_j \mid A \mid \phi_j\rangle = \mathrm{tr}[\rho A].$$

spectral resolution (when does it exist?)

$$\begin{aligned} P_i &:= \mid a_i \rangle\langle a_i \mid \\ A &= \sum_i a_i P_i = \sum_i a_i \mid a_i \rangle\langle a_i \mid \end{aligned}$$

3.2 Multiple-qubit operators

4 Entanglement

Definition 4.1. A **separable state** is a decomposable tensor; a **entangled state** is an indecomposable tensor.

[xxx rank notion, and examples]

[xxx triviality of determining entanglement in the finite-dimensional case. Only for rank-two tensors? More difficult for higher-rank tensor products?]

5 Quantum teleportation

teleportation. Alice and Bob. Draw up the figure.

Cite result [**NC**]: this can be done with maximum entanglement. It cannot be done ($p < 1$) without maximum entanglement. Include the proof.

6 Entanglement swapping

Open question: how to do entanglement swapping when input states are mixed.

6.1 Lattices

lattice

6.2 Triangles

draw the figure.

Such a thing can be constructed. **star-triangle formulation**. Cite.

Explicitly write down the preparation matrix.

Open question: to what use can such a thing be put?

A Tensor products and Kronecker notation

Keep it brief and practical. State (with don't-be-scared caveat) the abstract-algebra definition. Then immediately give examples showing that these are just pairs (or n-tuples) with the manipulation rules (which is the practical meaning of the equivalence relations) of scalar-through and multilinearity. These give different ways to write the same thing, and give us some flexibility for computations.

Lift stuff from prolrev.

decomposable tensors. xref both ways between this and entangled states in QM.

A.1 Tensor products and array notation

Tableaux for tensor product of vectors. Lift from prolrev.

A.2 Kronecker notation

Work out a quick 2×2 example of why this is the right thing to do.

A.3 Linear operators on tensor products

$$(A \otimes B)(\mathbf{u} \otimes \mathbf{v}) := (A\mathbf{u}) \otimes (B\mathbf{v})$$

work it out by expansion of eigenbases. Use $\mathbf{u}$ and $\mathbf{v}$ for this appx; also be sure to use u_i and v_j for coeffs.

Kronecker product

xxx:

$$i \Big\downarrow \begin{pmatrix} (A\mathbf{u})_1 \\ (A\mathbf{u})_2 \\ (A\mathbf{u})_3 \end{pmatrix} \quad = \quad i \Big\downarrow \overset{\displaystyle \overset{k}{\longrightarrow}}{\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}} \quad k \Big\downarrow \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

Mnemonic: adjacency of coefficients in 2nd argument of vkron, so B is blocked tighter.

$$\left(\begin{array}{cc|cc} A_{11}B_{11} & A_{11}B_{12} & A_{12}B_{11} & A_{12}B_{12} \\ A_{11}B_{21} & A_{11}B_{22} & A_{12}B_{21} & A_{12}B_{22} \\ \hline A_{21}B_{11} & A_{21}B_{12} & A_{22}B_{11} & A_{22}B_{12} \\ A_{21}B_{21} & A_{21}B_{22} & A_{22}B_{21} & A_{22}B_{22} \end{array}\right) \left(\begin{array}{c} u_1v_1 \\ u_1v_2 \\ \hline u_2v_1 \\ u_2v_2 \end{array}\right)$$

$$\left(\begin{array}{c|c} A_{11}B & A_{12}B \\ \hline A_{21}B & A_{22}B \end{array}\right) \left(\begin{array}{c} u_1\mathbf{v} \\ \hline u_2\mathbf{v} \end{array}\right)$$

A.4 Inner products on tensor products

xxx present this as a special case of the above:

$$\langle (\mathbf{a} \otimes \mathbf{b}), (\mathbf{c} \otimes \mathbf{d}) \rangle = \langle \mathbf{a}, \mathbf{c} \rangle \langle \mathbf{b}, \mathbf{d} \rangle.$$

B Trace and partial trace

do it in the kronecker rep too. 2×2 on A and B.

type up the handwritten notes.

I think left-partial is trace within blocks; right partial is sum of blocks.

Is this just

$$\mathrm{tr}_{\mathrm{left}} \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right) = \begin{pmatrix} \mathrm{tr}(A) & 0 \\ 0 & \mathrm{tr}(D) \end{pmatrix}$$

and

$$\mathrm{tr}_{\mathrm{right}} \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right) = A + D?$$

Work out some computational examples. And of course code it up. `:)`

C Density operators

Definition C.1. An **ensemble** is a list of n state vectors $\{\phi_1, \ldots, \phi_n\}$ along with respective probabilities $\{p_1, \ldots, p_n\}$ with $0 \leq p_i \leq 1$ and $\sum_{i=1}^{n} p_i = 1$.

One may think of an ensemble as a probability mass function.

Definition C.2. A **density matrix** is a positive-definite matrix with trace 1.

Proposition C.3. *An $n \times n$ density matrix may always be obtained from an ensemble by*

$$\rho = \sum_{i=1}^{n} p_i \mid \phi_i \rangle \langle \phi_i \mid .$$

Proof. xxx type it up. □

Proposition C.4. *An $n \times n$ density matrix may be obtained from a state ψ and a basis $\{\phi_1, \ldots, \phi_n\}$ by*

$$\rho = \sum_{i=1}^{n} p_i \mid \phi_i \rangle \langle \phi_i \mid$$

where

$$p_i = |\langle \psi \mid \phi_i \rangle|^2.$$

Proof. xxx type it up. □

Remark C.5. Two different ensembles can give the same density matrix. N&C give a theorem characterizing the conditions under which this can happen. [xxx type this up and include examples.]

Definition C.6. A state ψ is said to be a **pure state** with respect to an ensemble $\{\phi_1, \dots, \phi_n\}$ if it is equal to one of the ϕ_i's. Otherwise it is said to be a **mixed state**.

Remark C.7. Pure and mixed states have no meaning except with respect to a specified ensemble. A state that is pure with respect to one ensemble may be mixed with respect to another.

Proposition C.8. *A density matrix ρ has $\operatorname{tr}(\rho) = 1$. Furthermore, if ρ is a density matrix for a state ψ and a basis $\{\phi_1, \dots, \phi_n\}$ (as in the statement of proposition C.4), then $\operatorname{tr}(\rho^2) \le 1$, with equality if and only if ψ is pure with respect to the basis.*

Proof. For the duration of this proof, index the basis vectors as

$$\{\phi^{(1)}, \dots, \phi^{(n)}\}.$$

Then subscripts will denote elements of a vector. For example, if $\phi^{(1)} = (0.6, 0.8)$, then $\phi_2^{(1)} = 0.8$.

By proposition C.3, there is an ensemble such that

$$\rho = \sum_{k=1}^{n} p_k \mid \phi^{(k)} \rangle\langle \phi^{(k)} \mid .$$

Now, each $\mid \phi^{(k)} \rangle\langle \phi^{(k)} \mid$ is an outer-product matrix with ijth entry equal to

$$(\mid \phi^{(k)} \rangle\langle \phi^{(k)} \mid)_{ij} = \sum_{k=1}^{n} \phi_i^{(k)} \phi_j^{(k)*}.$$

Since the trace of an $n \times n$ matrix A is $\operatorname{tr}(A) = \sum_{i=1}^{n} A_{ii}$, we have

$$\begin{aligned}
\operatorname{tr}(\rho) &= \sum_{i=1}^{n} \rho_{ii} \\
&= \sum_{i=1}^{n} \sum_{k=1}^{n} p_k \phi_i^{(k)} \phi_i^{(k)*} \\
&= \sum_{k=1}^{n} p_k \sum_{i=1}^{n} |\phi_i^{(k)}|^2 \\
&= \sum_{k=1}^{n} p_k \|\phi^{(k)}\| \\
&= \sum_{k=1}^{n} p_k \\
&= 1.
\end{aligned}$$

For the second claim, first note that for an $n \times n$ matrix A, we have

$$\operatorname{tr}(A^2) = \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} A_{ji}.$$

Then

$$\begin{aligned}
\operatorname{tr}(\rho^2) &= \sum_{i=1}^{n}\sum_{j=1}^{n} \rho_{ij}\rho_{ji} \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} \left(\sum_{k=1}^{n} p_k \phi_i^{(k)} \phi_j^{(k)*}\right)\left(\sum_{\ell=1}^{n} p_\ell \phi_j^{(\ell)} \phi_i^{(\ell)*}\right) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{\ell=1}^{n} p_k p_\ell \phi_i^{(k)} \phi_j^{(k)*} \phi_j^{(\ell)} \phi_i^{(\ell)*} \\
&= \sum_{k=1}^{n} p_k \sum_{\ell=1}^{n} p_\ell \sum_{i=1}^{n} \phi_i^{(k)} \phi_i^{(\ell)*} \sum_{j=1}^{n} \phi_j^{(\ell)} \phi_j^{(k)*} \\
&= \sum_{k=1}^{n} p_k \sum_{\ell=1}^{n} p_\ell \langle \phi^{(k)} \mid \phi^{(\ell)} \rangle \langle \phi^{(\ell)} \mid \phi^{(k)} \rangle \\
&= \sum_{k=1}^{n} p_k \sum_{\ell=1}^{n} p_\ell |\langle \phi^{(k)} \mid \phi^{(\ell)} \rangle|^2.
\end{aligned}$$

Now, since I assume the basis is orthonormal, we have

$$\operatorname{tr}(\rho^2) = \sum_{k=1}^{n} p_k \sum_{\ell=1}^{n} p_\ell \delta_{k\ell} = \sum_{k=1}^{n} p_k^2.$$

To finish the proof, note that $\sum_{k=1}^{n} p_k^2$ is the diagonal part of $(\sum_{k=1}^{n} p_k)^2$. We have

$$\begin{aligned}
1 = (1)^2 &= \left(\sum_{k=1}^{n} p_k\right)^2 \\
&= \sum_{k=1}^{n}\sum_{\ell=1}^{n} p_k p_\ell \\
&= \sum_{k=1}^{n}\sum_{\ell \neq k} p_k p_\ell + \sum_{k=1}^{n} p_k^2.
\end{aligned}$$

If only one p_k is 1, then we clearly have equality. Note that all the terms in the sum are non-negative. If two p_k's are non-zero (say, p_1 and p_2) then the off-diagonal term is non-zero, so the diagonal sum is less than 1. □

Remark C.9. The trace $\operatorname{tr}(\rho)$ is basis-independent; $\operatorname{tr}(\rho^2)$ is basis-dependent. For example, let

$$\psi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

If

$$\phi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \phi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

then

$$\rho = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \rho^2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \operatorname{tr}(\rho) = 1, \quad \text{and} \quad \operatorname{tr}(\rho^2) = 1.$$

On the other hand, if

$$\phi_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \quad \text{and} \quad \phi_2 = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix},$$

then

$$\rho = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}, \quad \rho^2 = \begin{pmatrix} 1/4 & 0 \\ 0 & 1/4 \end{pmatrix}, \quad \mathrm{tr}(\rho) = 1, \quad \text{and} \quad \mathrm{tr}(\rho^2) = 0.5.$$

Remark C.10. In the finite-dimensional case, determining purity of a state is trivial: form the density matrix ρ and compute $\mathrm{tr}(\rho^2)$. This is 1 iff ψ is pure.

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