Notes for Lie groups

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Abstract

This is a crib sheet for Lie groups. It is under construction.

Contents

1 Introduction

The art of doing mathematics consists in finding that special case which contains all the germs of generality.

— David Hilbert (1862-1943).

Focus:

- Exemplar model
- Connections with vector calc whenever possible.
- Motivation and otherwise-unsolvable problems: why do we *need* Lie groups?

To do:

- As much diffeo magic as possible
- Go thru all Palmer's HW assignments.

2 Lie groups

Definition 2.1. A **Lie group** is a smooth manifold G with an operation on points which make it a group. Furthermore, group multiplication and inversion must be smooth maps from $G \times G \to G$ and $G \to G$, respectively.

Example 2.2. $\mathbb{R}^n$ with addition.

Std. examples: $\mathbb{R}^n$, projective spaces, spheres; finite groups w/ discrete topo?; GL, SU, SO, U, O, $Sp(n)$, $Spin(n)$.

Esp. $n = 1$ cases.

L and R maps.

Left-invariance.

Left-invt vr. flds.

Lie bracket.

abstract def'n of Lie alg. Mention cross products. Show that these algebras satisfy the definition.

Ad

diffeos. L_g and R_g.

integral curve.

1-parm sgr. $f : \mathbb{R} \to G$.

Exemplars: $\mathbb{R}^2$, $\mathbb{R}^3$, S^3, S^1, $SL(2, \mathbb{R})$.

3 Lie algebras

axiomatic def'n. emph. *not* an algebra (non-associative). Doesn't need to come from a Lie group. Example: $\mathbb{R}^3$ with cross product.

Lie algebras: two ways. (1) left-invt vr. fields; (2) tangent space at identity.

Lie algebras for std. examples. Make sure to explain WHY.

Esp. $O(n)$, $SO(n)$, $SU(n)$, $Spin(n)$.

bracket; mx $[A, B] = AB - BA$.

bij'n LG and T_eG p. 129.

4 Lie groups and their algebras

$G \to LG$; $g \to \ell g$.

exp; mx exp is e^A; exp vs. Φ.

$F : G_1 \to G_2$; $F_* : g_1 \to g_2$.

capitalization conventions.

Ad, ad, rep spaces. d/dt p. 130.

SO, O, SU, Sp; their L's. And dimensions.

p. 153 double cover.

$(L_g)_*$ and $(R_g)_*$ for mx groups.

5 Haar measure

6 Killing form

p. 172

7 The Cartan subalgebra

8 Roots and root spaces

9 Clifford algebras?

10 Complexification of Lie algebras

References

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