Induction and the sum of consecutive squares

John Kerl · Math 110, section 2 · Spring 2006

In chapter 5 we encountered formulas for the sum of consecutive integers and the sum of consecutive squares:

\[ \sum_{k=1}^{n} k = \frac{n(n+1)}{2} \quad \text{and} \quad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}. \]

For example,

\[ 1 + 2 + 3 + 4 + 5 = \frac{5 \cdot 6}{2} = 15 \quad \text{and} \quad 1 + 4 + 9 + 16 + 25 = \frac{5 \cdot 6 \cdot 11}{6} = 55. \]

But this story has a significant omission: Why are these true — are we just supposed to accept these formulas on faith? Does trying the formula out for a few values of \( n \) convince you it’s true for every \( n \)? This question is answered easily enough, using a technique called induction. This technique could be taught in college algebra (maybe at some universities it is), and it’s useful throughout mathematics.

1 Induction

The idea of an inductive proof is as follows: Suppose you want to show that something is true for all positive integers \( n \). (The catch: you have to already know what you want to prove — induction can prove a formula is true, but it won’t produce a formula you haven’t already guessed at.)

- **Step 0.** Come up with a formula, and give yourself reason to think it’s true.
- **Step 1** (base case). Show the formula holds for \( n = 1 \).
- **Step 2** (induction step). Suppose it’s true for \( n - 1 \), and then show it’s true for \( n \).

This is a leapfrog type of argument. It’s kind of like evaluating the terms of a sequence: you have a starting point, and you get the next term by doing something to the previous term. This chains along for as long as you have patience.

I showed in class (and your textbook also shows) why the first formula above, the sum-of-consecutive-integers formula, is true. (It’s an arithmetic sum.) So we could use induction for that formula, but we don’t need to. I’ll illustrate the induction technique by proving that the sum-of-consecutive-squares formula is true.

2 Sum of consecutive squares

Let’s see why

\[ \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}. \]
Step 0. How do we come up with the formula? In this case, it comes from your instructor and/or textbook authors. Does it look like it’s true? We can try a few values of \( n \). We did this in class, and above we had

\[
1 + 4 + 9 + 16 + 25 = \frac{5 \cdot 6 \cdot 11}{6} = 55.
\]

The formula claims that the sum should be 55, and when we add up the terms, we see it is 55.

Step 1. (Base case) Show the formula holds for \( n = 1 \). This is usually the easy part of an induction proof. Here, this is just

\[
\sum_{k=1}^{1} k^2 = 1 = \frac{(1+1)(2 \cdot 1 + 1)}{6} = \frac{1 \cdot 2 \cdot 3}{6} = 1.
\]

Step 2. (Induction step) Suppose it’s true for \( n - 1 \), and then show it’s true for \( n \). For this part, you usually need to do some algebraic manipulation. First we write down the expression for the sum of \( n \) consecutive squares:

\[
\sum_{k=1}^{n} k^2.
\]

We’re pretending we don’t know that this is \( n(n+1)(2n+1)/6 \), so we don’t give ourselves permission to write that down yet. Our job is to come up with that as a consequence of what we are sure of, namely, that it’s true for \( n - 1 \).

The trick is that this is just the sum of \( n \) numbers, so we can split it up into the sum of the first \( n - 1 \) terms, and the very last term all by itself:

\[
\sum_{k=1}^{n} k^2 = \sum_{k=1}^{n-1} k^2 + \sum_{k=n}^{n} k^2 = \sum_{k=1}^{n-1} k^2 + n^2.
\]

Now, we’re pretending we are confident of what the sum of the first \( n - 1 \) consecutive squares is, so we can write this as:

\[
\sum_{k=1}^{n} k^2 = \sum_{k=1}^{n-1} k^2 + n^2 = \frac{(n-1)(n-1+1)(2(n-1)+1)}{6} + n^2
\]

\[
= \frac{(n-1)(n)(2n-1)}{6} + n^2.
\]

We need to turn this into what we want, which is

\[
\frac{n(n+1)(2n+1)}{6}.
\]

The question then reduces to:

\[
\frac{(n-1)(n)(2n-1)}{6} + n^2 \equiv \frac{n(n+1)(2n+1)}{6}.
\]

All we need to do is FOIL these out and simplify. Since the right-hand side has one big denominator of 6, maybe we could put the left-hand side into the same form.

The left-hand side is

\[
\frac{(n-1)(n)(2n-1)}{6} + n^2 = \frac{(n-1)(n)(2n-1) + 6n^2}{6}
\]
which FOILs out to
\[ \frac{(n^2 - n)(2n - 1) + 6n^2}{6} = \frac{2n^3 - n^2 - 2n^2 + n + 6n^2}{6} = \frac{2n^3 + 3n^2 + n}{6}. \]

The right-hand side FOILs out to
\[ \frac{n(n + 1)(2n + 1)}{6} = \frac{(n^2 + n)(2n + 1)}{6} = \frac{2n^3 + n^2 + 2n^2 + n}{6} = \frac{2n^3 + 3n^2 + n}{6} \]

which is the same thing as the left-hand side.

In summary, we showed that the formula is true for \( n = 1 \). Then we showed that if the sum-of-consecutive-squares formula is true for an integer, then it’s also true for the next integer. Since it’s true for \( n = 1 \) (the base case), it’s true for \( n = 2 \) by the induction step. Since it’s true for \( n = 2 \), it’s true for \( n = 3 \) by the induction step, and so on. Then we are sure that it’s true for any \( n \) at all.

### 3 Two questions

Since we have formulas for the sum of consecutive integers and the sum of consecutive squares, it’s natural to wonder: what do you get when you sum up consecutive cubes? For example,
\[
\begin{align*}
1 &= 1 \\
1 + 8 &= 9 \\
1 + 8 + 27 &= 36 \\
\vdots \\
\end{align*}
\]

Here’s another one — the sum of consecutive odd integers:
\[
\begin{align*}
1 &= 1 \\
1 + 3 &= 4 \\
1 + 3 + 5 &= 9 \\
\vdots \\
\end{align*}
\]

For either of these: Do you see a pattern? Can you guess a formula? Can you convince yourself it’s probably true? Can you prove it’s true?

### 4 More information