Constructions of the Lebesgue integral

John Kerl

January 23, 2006

Abstract

Various constructions of the Lebesgue integral are compared and contrasted. This paper fulfills a term-paper requirement for Bill Faris' spring 2006 Real Analysis course (Math 523B).

This paper is under construction.

Contents

1 Introduction

Objectives for the paper:

- Make some brief mention of the history.
- Focus mainly on a survey of modern treatments, including terminology and order of presentation.

Objectives for the construction:

- Compatibility with Riemann integral — e.g. FTC, area, linearity
- Monotone convergence/countable additivity
- Ability to integrate functions the Riemann integral cannot; extending the concept of measure beyond integration.

Constructions:

- Riemann and Riemann-Stieltjes as background
- Original Lebesgue construction? He is quoted as saying he "partitioned the range rather than the domain", although the Daniell construction as presented by Faris does not work this way.
- Carathéodory
- Daniell
- Other?

Preconditions:

- Topological notions such as compactness of domain
- Under what conditions do we require the domain X to be merely a set? A topological space? A metric space?

My notes:

- Explain the reason for the terms *ring* and *algebra*. These are suggestive of the terms from abstract algebra. Why were these words chosen?
- Ring and algebra for sets as well as functions.

Faris notes:

- Given a pre-measure on a ring, or a pre-integral on a vector lattice, how you get a measure or an integral?

- How to show that an additive set function on a ring, or a linear function on a vector lattice, actually satisfies monotone convergence or countable additivity. Can it be done without compactness or some other topological notion?
- Catalog the terminologies used throughout the literature. What sources?
  - Rudin
  - Royden
  - Folland
  - Dudley
  - Who else?

  Have one pair of sections for the basics of the two constructions. Then, *separately*, catalog the terminologies.
- Always describe *reasons* for each definition and precondition: *why* do we define it this way?

2 Measures-first approach

- $\mu(0) = \emptyset$
- $E_i \cap E_j = \emptyset \implies \mu(\cup E_i) = \sum \mu(E_i)$.
- xlat'n invar't
- $\mu(Q) = 1$.

The last two are geometric; the first two are properties for general measures.

Monotonicity is a consequence: $E \subseteq F \implies \mu(E) \leq \mu(F)$.

Subadditivity is a consequence: $\mu(\cup E_i) \leq \sum \mu(E_i)$.

Set MCT, set DCT

To *construct*:

Def outer measure.

Then, prove properties are satisfied.

algebra to σ-algebra: def pre-measure: same first two props, but only on an algebra.

$\mu(g - f) = \mu(g) - \mu(f)$: by linearity. Order preservation: $g \geq f \implies \mu(g) \geq \mu(f)$, i.e. $g - f \geq 0 \implies \mu(g - f) \geq 0$: orientation?

3 Functions-first approach

We start with a pre-integral on a Stone vector lattice, then extend this to an integral on a σ-algebra. Definitions of these terms are as follows. (Please see appendix A for notation such as $f \wedge g$.)

Definition 3.1. Let X be a (set? topological space? metric space?). Let L be a set of functions from X to $\mathbb{R}$. Then L is a **Stone vector lattice** if the following conditions are satisfied:

- Vector space: If $f, g \in L$ and $a \in \mathbb{R}$, then $f + g, af \in L$.
- Lattice: If $f, g \in L$ then $f \wedge g, f \vee g \in L$.
- Stone property: If $f \in L$ then $f \wedge 1 \in L$. (Thus L has functions which are **locally constant**, without needing to have constant functions.)

Example 3.2. Take L to be the set of compactly supported continuous functions from $\mathbb{R}$ to $\mathbb{R}$.

Definition 3.3. Let $\mu : L \to \mathbb{R}$. Then μ is a **pre-integral** if the following conditions are satisfied:

- Linear: If $f, g \in L$ and $a \in \mathbb{R}$ then $\mu(f + g) = \mu(f) + \mu(g)$ and $\mu(af) = a\mu(f)$.
- Order-preserving: If $f < g$ then $\mu(f) < \mu(g)$.
- Monotone convergence: If $f_n \geq 0$ and $f_n \uparrow f$ then $\mu(f_n) \uparrow \mu(f)$.

Definition 3.4. A **σ-algebra** of functions $\mathcal{F}$ satisfies the following conditions:

- $\mathcal{F}$ is a Stone vector lattice which contains constant functions.
- $\mathcal{F}$ is closed under pointwise convergence.
- (Note: a σ-ring is the same except it needn't have constant functions.)

Definition 3.5. Let $\mu : \mathcal{F}^+ \to [0, +\infty]$. Then μ is an **integral** if the following conditions are satisfied:

- asdfadsr
- asdfadsr
- asdfadsr

4 To be filed

finite meas/int; σ-finite meas/int.

measure space

σ-ring and σ-algebra on *sets* and *functions*.

$$\sum f(x_i)\Delta x_i \text{ vs. } \sum y_i \mu(f^{-1} y_i)?$$

5 Terminologies

A Notation

Notation A.1. We write $\mathbb{N}$ for the non-negative integers, $\mathbb{Z}$ for the integers, $\mathbb{Q}$ for the rationals, and $\mathbb{R}$ for the reals.

Throughout this section, let X be a set. Let $a \in \mathbb{R}$ and $f, g : X \to \mathbb{R}$.

Definition A.2. The **characteristic function** on X is given by

$$1_X(x) = \begin{cases} 1 & x \in X \\ 0 & x \notin X \end{cases}$$

Note that some authors use a χ or K in place of 1.

Definition A.3. We define $f + g$ and af to be

$$(f+g)(x) = f(x) + g(x), \quad (af)(x) = af(x) \text{ for } x \in X.$$

Notation A.4. If

$$f(x) < g(x) \quad \text{for all } x \in X$$

then we write

$$f < g.$$

Likewise for $f \leq g$, $f > g$, and $f \geq g$.

Definition A.5. We define $f \wedge g$ to be the function

$$(f \wedge g)(x) = \inf\{f(x), g(x)\} \text{ for } x \in X.$$

Likewise, we write $f \vee g$ for the function

$$(f \vee g)(x) = \sup\{f(x), g(x)\} \text{ for } x \in X.$$

Mnemonic A.6. The second letter of *infimum* is n, which looks something like $\wedge$. The second letter of *supremum* is u, which looks something like $\vee$.

Notation A.7. We write f_n for a sequence of functions from X to $\mathbb{R}$, for $n = 0, 1, 2, \ldots$.

Notation A.8. Let f_n be a sequence of functions from X to $\mathbb{R}$. We write $f_n \uparrow f$ or $f_n \downarrow f$ if:

- For all n, $f_n \leq f_{n+1}$ or $f_n \geq f_{n+1}$, respectively.
- For all n, $f_n \leq f$ or $f_n \geq f$, respectively.
- f_n converges to f pointwise.

References

[**Dud**] Dudley, R.M. *Real Analysis and Probability.* Cambridge, 2002.

[**Far**] Faris, W. *Real Analysis.* http://math.arizona.edu/~faris/realweb/realS06.html

[**Fol**] Folland, G. *Real Analysis: Modern Techniques and Their Applications* (2nd ed.). Wiley-Interscience, 1999.

[**Jon**] Jones, F. *Lebesgue Integration on Euclidean Space.* Jones and Bartlett, 2001.

[**Roy**] Royden, H.L. *Real Analysis* (2nd ed.). MacMillan, 1968.

[**Rud**] Rudin, W. *Principles of Mathematical Analysis* (3rd ed.). McGraw-Hill, 1976.

Index