Constructions of the Lebesgue integral

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Abstract

Various constructions of the Lebesgue integral are compared and contrasted. This paper fulfills a term-paper requirement for Bill Faris’ spring 2006 Real Analysis course (Math 523B).

This paper is under construction.
1 Introduction

Objectives for the paper:

- Make some brief mention of the history.
- Focus mainly on a survey of modern treatments, including terminology and order of presentation.

Objectives for the construction:

- Compatibility with Riemann integral — e.g. FTC, area, linearity
- Monotone convergence/countable additivity
- Ability to integrate functions the Riemann integral cannot; extending the concept of measure beyond integration.

Constructions:

- Riemann and Riemann-Stieltjes as background
- Original Lebesgue construction? He is quoted as saying he “partitioned the range rather than the domain”, although the Daniell construction as presented by Faris does not work this way.
- Carathéodory
- Daniell
- Other?

Preconditions:

- Topological notions such as compactness of domain
- Under what conditions do we require the domain $X$ to be merely a set? A topological space? A metric space?

My notes:

- Explain the reason for the terms ring and algebra. These are suggestive of the terms from abstract algebra. Why were these words chosen?
- Ring and algebra for sets as well as functions.

Faris notes:

- Given a pre-measure on a ring, or a pre-integral on a vector lattice, how you get a measure or an integral?
• How to show that an additive set function on a ring, or a linear function on a vector lattice, actually satisfies monotone convergence or countable additivity. Can it be done without compactness or some other topological notion?

• Catalog the terminologies used throughout the literature. What sources?
  – Rudin
  – Royden
  – Folland
  – Dudley
  – Who else?

Have one pair of sections for the basics of the two constructions. Then, separately, catalog the terminologies.

• Always describe reasons for each definition and precondition: why do we define it this way?

2 Measures-first approach

- \( \mu(\emptyset) = 0 \)
- \( E_i \cap E_j = \emptyset \implies \mu(\bigcup E_i) = \sum \mu(E_i) \).
- xlat’n invar’t
- \( \mu(Q) = 1 \).

The last two are geometric; the first two are properties for general measures.

Monotonicity is a consequence: \( E \subseteq F \implies \mu(E) \leq \mu(F) \).

Subadditivity is a consequence: \( \mu(\bigcup E_i) \leq \sum \mu(E_i) \).

Set MCT, set DCT

To construct:

Def outer measure.

Then, prove properties are satisfied.

algebra to \( \sigma \)-algebra: def pre-measure: same first two props, but only on an algebra.

\( \mu(g - f) = \mu(g) - \mu(f) \): by linearity. Order preservation: \( g \geq f \implies \mu(g) \geq \mu(f) \), i.e. \( g - f \geq 0 \implies \mu(g - f) \geq 0 \): orientation?

3 Functions-first approach

We start with a pre-integral on a Stone vector lattice, then extend this to an integral on a \( \sigma \)-algebra. Definitions of these terms are as follows. (Please see appendix A for notation such as \( f \wedge g \).)
Definition 3.1. Let $X$ be a (set? topological space? metric space?). Let $L$ be a set of functions from $X$ to $\mathbb{R}$. Then $L$ is a **Stone vector lattice** if the following conditions are satisfied:

- Vector space: If $f, g \in L$ and $a \in \mathbb{R}$, then $f + g, af \in L$.
- Lattice: If $f, g \in L$ then $f \land g, f \lor g \in L$.
- Stone property: If $f \in L$ then $f \land 1 \in L$. (Thus $L$ has functions which are **locally constant**, without needing to have constant functions.)

Example 3.2. Take $L$ to be the set of compactly supported continuous functions from $\mathbb{R}$ to $\mathbb{R}$.

Definition 3.3. Let $\mu : L \to \mathbb{R}$. Then $\mu$ is a **pre-integral** if the following conditions are satisfied:

- Linear: If $f, g \in L$ and $a \in \mathbb{R}$ then $\mu(f + g) = \mu(f) + \mu(g)$ and $\mu(af) = a\mu(f)$.
- Order-preserving: If $f < g$ then $\mu(f) < \mu(g)$.
- Monotone convergence: If $f_n \geq 0$ and $f_n \uparrow f$ then $\mu(f_n) \uparrow \mu(f)$.

Definition 3.4. A **$\sigma$-algebra** of functions $\mathcal{F}$ satisfies the following conditions:

- $\mathcal{F}$ is a Stone vector lattice which contains constant functions.
- $\mathcal{F}$ is closed under pointwise convergence.
- (Note: a $\sigma$-ring is the same except it needn’t have constant functions.)

Definition 3.5. Let $\mu : \mathcal{F}^+ \to [0, +\infty]$. Then $\mu$ is an **integral** if the following conditions are satisfied:

- asdfadsr
- asdfadsr
- asdfadsr

4 To be filed

finite meas/int; $\sigma$-finite meas/int.

measure space

$\sigma$-ring and $\sigma$-algebra on sets and functions.

$$\sum f(x_i) \Delta x_i \text{ vs. } \sum y_i \mu(f^{-1}(y_i))$$

5 Terminologies
A Notation

Notation A.1. We write $\mathbb{N}$ for the non-negative integers, $\mathbb{Z}$ for the integers, $\mathbb{Q}$ for the rationals, and $\mathbb{R}$ for the reals.

Throughout this section, let $X$ be a set. Let $a \in \mathbb{R}$ and $f, g : X \to \mathbb{R}$.

Definition A.2. The characteristic function on $X$ is given by

$$1_X(x) = \begin{cases} 1 & x \in X \\ 0 & x \notin X \end{cases}$$

Note that some authors use a $\chi$ or $K$ in place of $1$.

Definition A.3. We define $f + g$ and $af$ to be

$$(f + g)(x) = f(x) + g(x), \quad (af)(x) = af(x) \text{ for } x \in X.$$  

Notation A.4. If

$$f(x) < g(x) \quad \text{for all } x \in X$$

then we write

$$f < g.$$  

Likewise for $f \leq g$, $f > g$, and $f \geq g$.

Definition A.5. We define $f \wedge g$ to be the function

$$(f \wedge g)(x) = \inf \{f(x), g(x)\} \text{ for } x \in X.$$  

Likewise, we write $f \vee g$ for the function

$$(f \vee g)(x) = \sup \{f(x), g(x)\} \text{ for } x \in X.$$  

Mnemonic A.6. The second letter of $\text{infimum}$ is $n$, which looks something like $\wedge$. The second letter of $\text{supremum}$ is $u$, which looks something like $\vee$.

Notation A.7. We write $f_n$ for a sequence of functions from $X$ to $\mathbb{R}$, for $n = 0, 1, 2, \ldots$.

Notation A.8. Let $f_n$ be a sequence of functions from $X$ to $\mathbb{R}$. We write $f_n \uparrow f$ or $f_n \downarrow f$ if:

- For all $n$, $f_n \leq f_{n+1}$ or $f_n \geq f_{n+1}$, respectively.
- For all $n$, $f_n \leq f$ or $f_n \geq f$, respectively.
- $f_n$ converges to $f$ pointwise.
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