# Notes for complex analysis

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**Abstract**

The following are notes to help me prepare for the Complex Analysis portion of the University of Arizona math department's Geometry/Topology qualifier in August 2006. It is a condensed selection of the first seven chapters of Churchill and Brown, with some worked problems. Please also see my qualifier-solutions document, which you should find nearby as `prolrevqual.pdf`.

*This paper is under construction.*

# Contents

# 1 Qualifier topics

Per the qual packet, the topics are chs. 1-7 of [**CB**]:

- elementary functions,
- power series,
- singularities,
- residues,
- contour integrals, and
- conformal mappings including linear fractional transformations.

The standard complex-analysis problems on previous quals are (1) improper real integrals, to be evaluated using complex contour integrals, and (2) problems involving fractional linear transformations and conformal maps. Such integral problems are found in sections 58-61 of [**CB**]; for the purposes of this paper, sections 1-57 of [**CB**] appear only as necessary preliminary material.

# 2 Preliminaries

**Proposition 2.1** (Triangle inequality)**.**

$$|u \pm v| \leq |u| + |v|.$$

**Proposition 2.2** (Reverse triangle inequality)**.**

$$|u \pm v| \geq ||u| - |v||.$$

**Definition 2.3.** A **domain** is a connected open set. A **region** is a domain with none, some, or all of its boundary points adjoined.

# 3 Spherical and hyperbolic trigonometric functions

These functions are popular on the qualifying exams. Here are some necessary facts.

The key to this particular castle is the following foursome:

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2} \qquad \cosh(z) = \frac{e^z + e^{-z}}{2} \tag{3.1}$$

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i} \qquad \sinh(z) = \frac{e^z - e^{-z}}{2}. \tag{3.2}$$

From these we can obtain the Pythagorean identities and the sum and difference formulas:

$$\cos^2(\alpha) + \sin^2(\alpha) = 1$$
$$\cosh^2(\alpha) - \sinh^2(\alpha) = 1$$

$$\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta) \qquad \sinh(\alpha \pm \beta) = \sinh(\alpha)\cosh(\beta) \pm \cosh(\alpha)\sinh(\beta)$$
$$\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta) \qquad \cosh(\alpha \pm \beta) = \cosh(\alpha)\cosh(\beta) \pm \sinh(\alpha)\sinh(\beta).$$

Equations 3.1 and 3.2 also give

$$\sinh(iz) = i\sin(z) \qquad \sin(iz) = i\sinh(z) \tag{3.3}$$
$$\cosh(iz) = \cos(z) \qquad \cos(iz) = \cosh(z) \tag{3.4}$$

from which it follows that the spherical and hyperbolic trig functions are really the same on the complex plane, up to pre- and post-rotation by $\pi/2$.

**Mnemonic 3.1.** Just remember that the $i$'s behave just like the minus signs do in the familiar parity equations

$$\sinh(-x) = -\sinh(x) \qquad \sin(-x) = -\sin(x)$$
$$\cosh(-x) = \cosh(x) \qquad \cos(-x) = \cos(x)$$

except that you also add or remove an $h$ in all cases.

Lastly, from equations 3.3 and 3.4 and the sum and difference formulas we can get the complex sine and cosine in terms of the corresponding real functions:

$$\sin(x+iy) = \sin(x)\cos(iy) + \cos(x)\sin(iy) = \sin(x)\cosh(y) + i\cos(x)\sinh(y) \tag{3.5}$$
$$\cos(x+iy) = \cos(x)\cos(iy) - \sin(x)\sin(iy) = \cos(x)\cosh(y) - i\sin(x)\sinh(y). \tag{3.6}$$

Key point: remember equations 3.1 and 3.2 and you can derive the rest of these from scratch, without taking too much time on an exam. However, it doesn't hurt to memorize equations 3.5 and 3.6.

# 4 The point at infinity

**Definition 4.1.** The **extended complex plane**, written $\hat{\mathbb{C}}$, is the one-point compactification of $\mathbb{C}$. Center the unit 2-sphere $S^2$ (called the **Riemann sphere** in this context) on the complex plane so that the equator is on the unit circle. For each point $q$ on $S^2$ other than the north pole $N$, draw a line from $N$ to $q$ and find the intersection point $z$ of the line with the complex plane. This *stereographic projection* maps $\mathbb{C}$ bijectively with $S^2 \setminus \{N\}$. Now identify $N$ with the symbol $\infty$. (I.e. the entire horizon is treated as a *single point* at infinity. All outward paths lead to the same infinity.) Then $\mathbb{C} \cup \{\infty\}$ is $\hat{\mathbb{C}}$.

Here are two analysis-related notions involving the point at infinity:

**Definition 4.2.** The $\varepsilon$-ball around infinity is

$$B_\varepsilon(\infty) = \{z \in \hat{\mathbb{C}} : |z| > 1/\varepsilon\} = \{z \in \hat{\mathbb{C}} : 1/|z| < \varepsilon\}.$$

**Definition 4.3.**

$$\lim_{z\to\infty} = w_0 \iff \lim_{z\to 0} f(1/z) = w_0.$$

**Example 4.4.** $\triangleright$ Let

$$f(z) = \frac{az+b}{cz+d}.$$

Then

$$f(0) = b/d$$

whereas

$$\begin{aligned} f(\infty) &= \lim_{z \to 0} \frac{a/z + b}{c/z + d} \\ &= \lim_{z \to 0} \frac{a + bz}{c + dz} \\ &= a/c. \end{aligned}$$

See also section 12. ◁

# 5 Continuity, differentiability, and analyticity

Let $f : \mathbb{C} \to \mathbb{C}$.

**Definition 5.1.** Continuity is taken in the elementary sense: $f$ is **continuous** at $z_0$ if

- $\lim_{z \to z_0} f(z)$ exists, and
- $f(z_0)$ exists, and
- the two are equal.

**Definition 5.2.** Differentiability is taken in the elementary sense: $f$ is **differentiable** at $z_0$ if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists.

On the real line, there are only two ways for $x$ to approach $x_0$: from the left, and from the right. In the complex plane, there are infinitely many paths. Fortunately, we have a criterion for differentiability.

**Notation 5.3.** Write $w = f(z)$ with $w = u + iv$ and $z = x + iy$. Then $w = u(x, y) + iv(x, y)$ where $u, v : \mathbb{R}^2 \to \mathbb{R}$. Write $u_x$ for $\partial u/\partial x$ and likewise for $u_y$, $v_x$, and $v_y$.

**Proposition 5.4** (Cauchy-Riemann equations)**.** *If $u_x = v_y$ and $u_y = -v_x$ at $z_0$, then $f$ is differentiable at $z_0$. If $u_x$, $u_y$, $v_x$, and $v_y$ are continuous, then the converse is true as well.*

**Example 5.5.** The function $f(z) = z^2 = (x + iy)^2$ is differentiable everywhere by Cauchy-Riemann; the function $f(z) = |z|^2 = x^2 + y^2$ is differentiable only at zero. This is perhaps counterintuitive: these two functions are identical when restricted to the real line, where they are a nice, familiar parabola.

**Definition 5.6.** Note that we defined differentiability at a *point*. If $f$ is differentiable on an open set $U$, then $f$ is said to be **holomorphic** on $U$. If $f$ is holomorphic on all of $\mathbb{C}$, then $f$ is said to be **entire**.

**Remark 5.7.** Churchill and Brown say that $f$ is **analytic** if it is holomorphic in the above sense. (Since I am following Churchill and Brown, I will use the term *analytic* in place of *holomorphic* after this.) There is a technical distinction (see the Wikipedia [**Wik**] article on *holomorphic* for details):

- A (real or complex) function is said to be (real or complex) *analytic* if it is locally expressible as a convergent power series, i.e. has a Taylor-series expansion at each point. This requires the function to be infinitely differentiable (smooth).

- A *real* function can be merely differentiable without being smooth: One can devise functions which are $n$ times differentiable without being $n+1$ times differentiable. Furthermore, a function may be smooth without being real analytic. (The standard example [**Wik**] is $e^{-1/x^2}$ with value 0 filled in at 0.) Thus, there are proper containments: continuous ($C^0$) functions, differentiable ($C^1$) functions, twice-differentiable ($C^2$) functions, ..., smooth ($C^\infty$) functions, and real analytic function.
- For *complex* functions, though, these classes of functions are all the same (which justifies Churchill and Brown's use of the term *analytic* several chapters before defining power series): we will see in theorem 6.22 that a complex function which is once differentiable on an open set is necessarily infinitely differentiable on that open set, and is analytic in the power-series sense.

I believe **harmonic conjugates** are not needed for the qual, so I will not discuss them.

**Definition 5.8.** A function $f$ is said to be **singular** at $z_0$ if it is not analytic at $z_0$ but is analytic at *some* point in every neighborhood of $z_0$.

**Definition 5.9.** A function $f$ is said to have an **isolated singular point** at $z_0$ if $f$ is singular at $z_0$, and moreover there is a deleted neighborhood $0 < |z - z_0| < \varepsilon$ such that $f$ is analytic at *every* point in the neighborhood.

**Example 5.10.** Take $f(z) = 1/z$. This has an isolated singular point at 0.

# 6 Paths and integration

**Definition 6.1.** Let $w : [a, b] \to \mathbb{C}$, with $w = u + iv$ as above. (I suppose we can call such a general $w$ a **path**, although we will use other words when $w$ satisfies additional criteria.) Then define the integral of a complex-valued function $w(t)$ of a real variable to be

$$\int_a^b w(t)dt = \int_a^b u(x(t), y(t))dt + i \int_a^b v(x(t), y(t))dt.$$

Likewise, $w$ has derivative

$$w'(t) = u'(t) + iv'(t).$$

**Proposition 6.2.**

$$\left| \int_a^b w(t)dt \right| \leq \int_a^b |w(t)|dt.$$

This is well and good, but we want to be able to integrate complex-valued functions $f(z)$ of a *complex* variable as well. Real integrals are defined for $t$ on line segments; in the complex plane, we have another degree of freedom of motion for $z$ and we need to define the notion of path.

**Definition 6.3.** Here are some kinds of paths:

- An **arc** $z(t) = x(t) + iy(t)$ is a path with $x(t)$ and $y(t)$ being continuous functions on $t \in [a, b]$.
- A **simple** arc is one that does not cross itself, except perhaps at the endpoints: $t_1 \neq t_2$ and $t_1, t_2 \neq a, b$ implies $w(t_1) \neq w(t_2)$.
- A **closed curve** is an arc for which $w(a) = w(b)$.
- A **smooth arc** has a continuous tangent vector.

- A **contour** is a piecewise smooth arc, with finitely many pieces.

**Remark 6.4.** Churchill and Brown here refer to the *Jordan curve theorem*, which is intuitively undeniable: a simple closed curve separates $C$ into a bounded interior and an unbounded exterior.

**Intuition 6.5.** Perhaps the only "definition" you need for contours is the following: all the kinds of paths we normally think to integrate over — line segments, semicircles, etc. — are contours.

**Definition 6.6.** Now we can define a *contour integral*. This could be defined as a Riemann sum, or by making use of definition 6.1:

$$\oint_C f(z)dz = \int_a^b f(z(t))\frac{dz}{dt}dt.$$

We immediately wonder if and when the value of the integral depends on the path taken.

**Proposition 6.7.** *Let $f$ be continuous on a domain $D$. The following are equivalent:*

- *$f$ has an antiderivative $F$ on $D$;*
- *integrals along not-necessarily-closed paths in $D$ are independent of path;*
- *integrals around closed contours in $D$ are zero.*

**Remark 6.8.** The hypothesis is continuity, not analyticity.

**Remark 6.9.** This means that interesting contour integrals will need to have at least a discontinuity within the contour.

**Theorem 6.10** (ML estimate)**.** *Let $C$ be a contour, of arc length $L$. Furthermore suppose that $|f(z)| \leq M$ for all $z \in C$. Then*

$$\left|\int_C f(z)dz\right| \leq ML.$$

*Proof (David Herzog).* Let $z(t)$ be a parameterization of the contour $C$ for $t \in [0,1]$. Then

$$\int_C f(z)dz = \int_0^1 f(z(t))\frac{dz}{dt}dt.$$

Since $|f(z)| \leq M$ on $C$,

$$\begin{aligned}\left|\int_C f(z)dz\right| &\leq \int_0^1 |f(z(t))|\,|z'(t)|\,dt \\ &\leq M\int_0^1 |z'(t)|\,dt \\ &\leq ML.\end{aligned}$$

□

**Remark 6.11.** This theorem is not profound, but it is crucial for qual-type calculations. See section 10.

**Theorem 6.12** (Maximum modulus principle)**.** *If a function $f$ is analytic and not constant in a given domain $D$, then $|f(z)|$ has no maximum value in $D$.*

**Theorem 6.13** (Jordan's inequality)**.** *Let $C_R$ be the semicircle of radius $R$ given by $Re^{i\theta}$ for $\theta$ from 0 to $\pi$. Suppose $q(z)$ is a rational function with all its upper-half-plane zeroes below $C_R$. If there is an $M_R$ such that $|q(z)| \le M_R$ for all $z \in C_R$ and such that $M_R \to 0$ as $R \to \infty$, then for all positive real $a$,*

$$\int_{C_R} q(z)e^{iaz}dz \to 0 \quad as \quad R \to \infty.$$

**Theorem 6.14** (Jordan's lemma)**.** *Let $C_R$ be the semicircle of radius $R$ given by $Re^{i\theta}$ for $\theta$ from 0 to $\pi$. Suppose $f(z)$ is a function which is analytic on all of the upper half-plane exterior to the circle $|z| = R_0$. If there is an $M_R$ such that $|q(z)| \le M_R$ for all $z \in C_R$ and such that $M_R \to 0$ as $R \to \infty$, then for all positive real $a$,*

$$\int_{C_R} f(z)e^{iaz}dz \to 0 \quad as \quad R \to \infty.$$

**Theorem 6.15** (Cauchy-Goursat)**.** *If $f$ is analytic on and in a simple closed contour $C$, then*

$$\oint_C f(z)dz = 0.$$

Here is an extension:

**Proposition 6.16.** *If $f$ is analytic throughout a simply connected domain $D$, then for all closed contours $C$ (not necessarily simple closed contours),*

$$\oint_C f(z)dz = 0.$$

**Corollary 6.17.** *If $f$ is analytic on a simply connected domain $D$, then $f$ has an antiderivative on $D$.*

**Theorem 6.18** (Cauchy integral formula)**.** *If $f$ is analytic on and in a simple closed contour $C$, and if $z_0$ is in the interior of $C$, then*

$$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)dz}{z - z_0}.$$

**Remark 6.19.** This differs from Cauchy-Goursat in that a discontiunity has intentionally been inserted via dividing by $z - z_0$.

**Corollary 6.20** (Gauss mean value)**.** *If $f$ is analytic in and on the circle of radius $\rho$ centered at $z_0$, then*

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + \rho e^{i\theta})d\theta.$$

*Proof.* In the Cauchy integral formula, put $z = z_0 + \rho e^{i\theta}$. Then $dz = \rho i e^{i\theta}d\theta$. Write it down and simplify, and you're done. □

**Proposition 6.21** (Deformation of paths)**.** *If $C_2$ is inside $C_1$, and if $f$ is analytic in the intermediate region, then*

$$\oint_{C_1} f(z)dz = \oint_{C_2} f(z)dz = \oint_C f(z)dz$$

*for all $C$ in that intermediate region.*

**Theorem 6.22.** *If $f$ is analytic at a point, then it is infinitely differentiable and all its derivatives are analytic. Specifically,*

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)dz}{(z - z_0)^{n+1}}.$$

*In particular,*

$$f'(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)dz}{(z - z_0)^2}.$$

*Sketch of proof.* xxx fill in some details here. First do $f'$ using continuity of $f$: approximate $f(z)$ by $f(z_0)$ and use the ML estimate (6.10). Then, use induction for higher derivatives. ☐

**Remark 6.23.** This generalizes the Cauchy integral formula (theorem 6.18).

**Theorem 6.24** (Morera)**.** *(Converse to Cauchy-Goursat if $D$ is simply connected.) If $f$ is continuous on a domain $D$ and if $\oint_C f(z)dz = 0$ for all closed contours $C$ contained in $D$, then $f$ is analytic on $D$.*

**Theorem 6.25** (Liouville)**.** *If $f$ is bounded on all of $\mathbb{C}$ and if $f$ is entire, then $f$ is constant.*

# 7 Power series

**Theorem 7.1** (Taylor series)**.** *Let $f$ be analytic in the open disk*

$$D = \{z \in \mathbb{C} : |z - z_0| < R\}.$$

*Then for all $z \in D$, $f$ has the power series representation*

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)(z - z_0)^n}{n!}$$

**Remark 7.2.** If $f$ is entire then the radius $R$ may be taken arbitrarily large.

**Remark 7.3.** Recall from theorem 6.22 that for analytic $f$ we also have

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(s)ds}{(s - z_0)^{n+1}}.$$

Combining these two facts, we have

$$f(z) = \sum_{n=0}^{\infty} \left( \frac{1}{2\pi i} \oint_C \frac{f(s)ds}{(s - z_0)^{n+1}} \right) (z - z_0)^n.$$

**Mnemonic 7.4.** If we "cancel" powers of $(z - z_0)$ outside the sum with powers of $(s - z_0)$ inside the sum, then we get the same exponent as in the Cauchy integral formula (theorem 6.18).

Note that in order to use a Taylor expansion, we need $f$ to be analytic on an open disk. What if we have a function that is somehow singular, e.g. if $f$ itself is analytic but we consider $f(z)/(z - z_0)$?

**Theorem 7.5** (Laurent series)**.** *Let $f$ be analytic in the annulus*

$$A = \{z \in \mathbb{C} : R < |z - z_0| < S\}.$$

*Then for all $z \in A$, $f$ has the power series representation*

$$f(z) = \sum_{n=-\infty}^{\infty} \left( \frac{1}{2\pi i} \oint_C \frac{f(s)ds}{(s - z_0)^{n+1}} \right) (z - z_0)^n.$$

**Remark 7.6.** If $f$ is analytic in the disk (i.e. in the filled annulus) then one would expect the Laurent series to reduce to a Taylor series. Indeed this is the case: for analytic $f$ and negative $n$,

$$\frac{f(z)}{(z-z_0)^{n+1}} = f(z)(z-z_0)^{-n-1}$$

is the product of two analytic functions, hence analytic, so by Cauchy-Goursat the coefficients for negative $n$ are indeed zero.

Once we have a power series, we immediately want to be able to integrate and differentiate term by term. For the former, we have a stronger result. First, though, we need a definition.

**Definition 7.7.** Every power series

$$S(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

has a (perhaps infinite, perhaps zero-radius) **circle of convergence** centered at $z_0$:

- $S$ is (absolutely) convergent for all $z$ interior to $S$.
- $S$ is divergent for all $z$ exterior to $S$.

**Proposition 7.8.** *Let*

$$S(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

*be a power series. Let $C$ be a contour interior to the circle of convergence of $S$. Let $g(z)$ be any function that is continuous on $C$. Then $g(z)S(z)$ may be integrated term by term along $C$. That is,*

$$\int_C g(z) \left( \sum_{n=0}^{\infty} a_n (z-z_0)^n \right) dz = \sum_{n=0}^{\infty} a_n \int_C g(z)(z-z_0)^n dz.$$

**Proposition 7.9.** *Let*

$$S(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

*be a power series. Then $S$ may be differentiated term by term, i.e.*

$$S'(z) = \sum_{n=1}^{\infty} n a_n (z-z_0)^{n-1},$$

*anywhere inside its circle of convergence.*

# 8 Zeroes and poles

**Definition 8.1.** Let $f$ be analytic in an annulus $A$ centered at $z_0$. Then $f$ has Laurent series expansion for $z \in A$.

- The **principal part** of $f$ is the series with only negative-$n$ coefficients included.
- Let $m$ be index of the lowest-index non-zero coefficient in the Laurent series, if any.

- If there is no such $m$, then $f$ is said to have an **essential singularity** at $z_0$. (For example, write out the Taylor series for $e^z$ about $z_0 = 0$, then replace $z$'s with $1/z$'s to get a Taylor series for $e^{1/z}$.)
- If $m$ is negative, $f$ is said to have a **pole** of **order** $-m$ at $z_0$. (For example, $f(z) = 1/z^3$ at $z_0 = 0$.)
- If $m$ is positive, $f$ is said to have a **zero** of **order** $m$ at $z_0$. (For example, $f(z) = z^3$ at $z_0 = 0$.)
- A pole of order 1 is called a **simple pole**.
- A zero of order 1 is called a **simple zero**.
- If $m$ exists and is non-negative but $f$ is singular at $z_0$ (see definition 5.8), then $f$ is said to have a **removable singularity** at $z_0$. (For example, take $\sin(z)/z$. Write out the Taylor series for $\sin(z)$, then divide through by $z$: you will see no $z$'s in any denominator.)

**Definition 8.2.** A complex-valued function is said to be **meromorphic** or **regular** on an open subset $U$ of $\mathbb{C}$ if it is holomorphic on all of $U$ except for a set of isolated poles.

**Remark 8.3.** A meromorphic function may be written as the ratio of holomorphic functions, with denominator not identically zero. The canonical example is rational functions. Since polynomials are entire, rational functions are meromorphic on all of $\mathbb{C}$.

**Mnemonic 8.4.** Silly but memorable: *mero is ratio of holo.*

# 9 Residues

If $f$ has an isolated singular point $z_0$ (definition 5.9), then it is analytic in some deleted open neighborhood of $D$ $z_0$. Thus it has a Laurent-series expansion about $z_0$.

**Definition 9.1.** Let

$$\sum_{n=-\infty}^{\infty} a_k (z - z_0)^n$$

be the Laurent series expansion of $f$ on $D$. The **residue** of $f$ at $z_0$ is the coefficient on $1/(z - z_0)$ in the Laurent series expansion of $f$:

$$\operatorname*{Res}_{z=z_0} f(z) = a_{-1}.$$

By theorem 7.5 we see that this is the same as

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{2\pi i} \oint_C f(z) dz$$

for a positively oriented simple closed contour $C$ lying in $D$. By Cauchy-Goursat (theorem 6.15), if $f$ is analytic at $z_0$ it has zero residue there.

For poles (i.e. non-essential singularities), we have the following alternative formulas.

**Proposition 9.2.** *If $f(z)$ can be written as*

$$f(z) = \frac{\phi(z)}{z - z_0}$$

*with $\phi$ analytic and non-zero at $z_0$ (e.g. if $f$ has a zero of order 1 at $z_0$) then*

$$\operatorname*{Res}_{z=z_0} f(z) = \phi(z_0).$$

*If $f(z)$ can be written as*

$$f(z) = \frac{\phi(z)}{(z - z_0)^m}$$

*with $\phi$ analytic and non-zero at $z_0$ (e.g. if $f$ has a zero of order $m$ at $z_0$) then*

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}.$$

*If $f(z)$ can be written as*

$$f(z) = \frac{g(z)}{h(z)}$$

*where $g$ is analytic at $z_0$ and $h$ has a simple zero at $z_0$, then*

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{g(z_0)}{h'(z_0)}.$$

**Example 9.3.** $\triangleright$ The choice of formula can be significant, when one is pressed for time. Let

$$f(z) = \frac{z^{1/2}}{z^3 + 1},$$

taking the principal branch of the square root, and suppose we want to compute the sum of residues over poles of $f(z)$. Note that the denominator has the three roots [xxx include figure here]

$$\begin{aligned} z_1 &= -1 \\ z_2 &= \zeta_6 = e^{i\pi/3} = 1/2 + i\sqrt{3}/2 \\ z_3 &= \overline{\zeta}_6 = e^{-i\pi/3} = 1/2 - i\sqrt{3}/2. \end{aligned}$$

Using the first formula, we have

$$\begin{aligned} \operatorname*{Res}_{z=z_1} f(z) &= \frac{z_1^{1/2}}{(z_1 - z_2)(z_1 - z_3)} \\ \operatorname*{Res}_{z=z_2} f(z) &= \frac{z_2^{1/2}}{(z_2 - z_1)(z_2 - z_3)} \\ \operatorname*{Res}_{z=z_3} f(z) &= \frac{z_3^{1/2}}{(z_3 - z_1)(z_3 - z_2)}. \end{aligned}$$

You can get out your calculator and find that these are (respectively)

$$i/3, \quad -i/3, \quad i/3.$$

Doing it by hand, though, gives several opportunities for arithmetic errors.

Using the second formula, with $g(z) = z^{1/2}$ and $h(z) = z^3 + 1$, so $h'(z) = 3z^2$, we have

$$\begin{aligned} \operatorname*{Res}_{z=z_1} f(z) &= \frac{z_1^{1/2}}{3z_1^2} = \frac{1}{3z_1^{3/2}} \\ \operatorname*{Res}_{z=z_2} f(z) &= \frac{z_2^{1/2}}{3z_2^2} = \frac{1}{3z_2^{3/2}} \\ \operatorname*{Res}_{z=z_3} f(z) &= \frac{z_3^{1/2}}{3z_3^2} = \frac{1}{3z_3^{3/2}}. \end{aligned}$$

There's no subtraction going on, and trivial division, so the computation is less error-prone. The only catch is keeping track of the branches of the square root function. [xxx figure here.]

$$\begin{aligned} z_1^{3/2} &= i \\ z_2^{3/2} &= -i \\ z_3^{3/2} &= i \end{aligned}$$

so the sum of residues is

$$i/3 - i/3 + i/3 = i/3.$$

$\triangleleft$

**Theorem 9.4** (Residue theorem). *Let $C$ be a positively oriented simple closed contour. If $f$ is analytic on and in $C$ except at a finite number of singular points $z_k$ for $k = 1, 2, \ldots, n$ interior to $C$, then*

$$\frac{1}{2\pi i} \oint_C f(z) dz = \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z).$$

**Theorem 9.5** (Cauchy's argument principle). *Let $C$ be a simple closed curve. Let $f(z)$ be a function which is meromorphic in and on $C$, with no zeroes or poles on the boundary $C$. Let $N$ and $P$ be the number of zeroes and poles, respectively, of $f$ interior to $C$, counted with multiplicity. Then*

$$\oint_C \frac{f'(z)}{f(z)} dz = 2\pi i(N - P).$$

# 10 Integration

Now we come to the reason for everything up to this point: to use residues and contours to evaluate certain *improper real integrals.* We use the residue theorem (theorem 9.4) and the residue formulas in proposition 9.2. The canonical example is show that $\int_{-\infty}^{+\infty} \frac{\sin(x)}{x} dx = \pi$.

First a definition which often arises in real integrals. What do we mean by $\int_{-\infty}^{+\infty} f(x)dx$ — precisely *how* do the left and right endpoints go to their respective infinities?

**Definition 10.1.** The **Cauchy principal value** of the improper integral

$$\int_{-\infty}^{+\infty} f(x)dx,$$

written with a dash through the integral sign (for which I lack a pre-made LaTeX symbol), is

$$\lim_{a\to\infty} \int_a^a f(x)dx.$$

The need for this definition is best illustrated by example.

**Example 10.2.** $\triangleright$ Let $f(x) = x$. One might think that $\int_{-\infty}^{+\infty} f(x)dx$ should be zero — after all, doesn't the negative area on the left cancel the positive area on the right? Well, it does if you let the left and right edges go to zero at precisely the same rate. But this setup is fragile — you get something non-zero if you tweak it in some reasonable ways. First,

$$\lim_{a\to\infty} \int_a^{2a} f(x)dx$$

looks graphically like it shoots off to $+\infty$. Likewise,

$$\lim_{a\to\infty}\int_{2a}^{a} f(x)dx$$

looks like $-\infty$. The other tweak is **translation invariance**. Namely,

$$\int_{-\infty}^{+\infty} f(x)dx$$

ought to be the same as

$$\int_{-\infty}^{+\infty} f(x+c)dx.$$

Shift $f(x)=x$ by a constant and you'll see that the Cauchy principal value isn't zero.

My point here is that absolute integrability (which $f(x)=x$ lacks) is a necessary condition for a well-defined integral — one which is translation invariant and independent of parameterization of endpoints. The Cauchy principal value is best thought of as a *consolation prize* when a true integral is not possible. $\lhd$

**Remark 10.3.** If the integrand $f(x)$ is even, then

$$\int_{-\infty}^{+\infty} f(x)dx = 2\int_{0}^{+\infty} f(x)dx.$$

rat'l fcns, sin/cos times rat'l, sin/cos definite int'l, thru branch cut

Use $z^{1/2}/(1+z^3)$ as an example:

- Two different ways to compute residues (one better).
- Keyhole path. $C_1$ is the desired; $C_2$ is circle with $R\to\infty$; $C_3$ appears to be negative of desired but is not due to the square-root branch; $C_4$ is circle the other way with $r\to 0$.

  $C_3$:

  $$\int_r^R \frac{x^{1/2}}{x^3+1}dx$$

  but here [xxx figure] $x^{1/2}=-x$ rather than $-\sqrt{x}$. So

  $$\int_R^r \frac{-\sqrt{x}}{x^3+1}dx = \int_r^R \frac{-\sqrt{x}}{x^3+1}dx.$$

- Define the branch of the square root: $Re^{i\theta}\mapsto \sqrt{R}e^{i\theta/2}$ for $0\le\theta<2\pi$.
- Note easy numerical check for the integral. Also graph the integrand.
- How to estimate integral on inner and outer circles? For small $z$, $f(z)\approx z^{1/2}/1\to 0$; for large $z$, $f(z)\approx z^{1/2}/z^3\to 0$. Then couple that with an ML estimate.

## 11 Conformal maps

**Definition 11.1.** A function $f : \mathbb{C} \to \mathbb{C}$ is **conformal** at a point $z_0$ if it is analytic at $z_0$ and $f'(z_0) \neq 0$. Likewise, it is conformal on a domain $D$ if it is analytic at each point of $D$.

**Proposition 11.2.** *Conformal maps are angle-preserving.*

*Proof.* (This discussion is adapted from [**CB**].) Let $z_1(t)$, $z_2(t)$ be curves intersecting at a common point $z_0$ at time $t_0$, i.e. $z_1(t_0) = z_2(t_0) = z_0$. Let $w_1(t) = f(z_1(t))$ and $w_2(t) = f(z_2(t))$. The key insight is that the angle between curves is (up to $2\pi$) the difference of the complex phases of their derivatives. [xxx include figure here.] Using the chain rule, we have

$$w_1'(t) = f'(z_1(t))z_1'(t) \qquad \text{and} \qquad w_2'(t) = f'(z_2(t))z_2'(t).$$

When we multiply two complex numbers , their phases add. So,

$$\begin{aligned} \operatorname{Arg}(w_1'(t)) &= \operatorname{Arg}(f'(z_1(t))) + \operatorname{Arg}(z_1'(t)) \\ \operatorname{Arg}(w_2'(t)) &= \operatorname{Arg}(f'(z_2(t))) + \operatorname{Arg}(z_2'(t)). \end{aligned}$$

But the two curves intersect at $t_0$, so when $t = t_0$, the first terms after the equals signs are the same. Subtracting, we have

$$\operatorname{Arg}(w_1'(t_0)) - \operatorname{Arg}(w_2'(t_0)) = \operatorname{Arg}(z_1'(t)) - \operatorname{Arg}(z_2'(t)).$$

That is, the difference in phases of the image curves is the same as the difference in phases of the original curves, which is what we mean by angle-preserving. □

**Example 11.3.** ▷ Let $z_1(t) = t$ and $z_2(t) = it$, with $t_0 = 0$. Let $f(z) = e^z$. Then

$$f'(z_i(t_0)) = f'(0) = e^0 = 1$$

$$\begin{aligned} z_1'(t_0) &= 1 & z_2'(t) &= i \\ w_1(t) &= e^t & w_2(t) &= e^{it} \\ w_1'(t) &= e^t & w_2'(t) &= ie^{it} \end{aligned}$$

so

$$w_1'(0) = 1 \quad \text{and} \quad w_2'(t) = i.$$

The input phases are 0 and $\pi/2$ respectively, as are (in this trivial example) the output phases. ◁

**Proposition 11.4.** *Conformal maps preserve fundamental groups.*

**Theorem 11.5** (Riemann mapping theorem)**.** *Let $U$ be a simply connected open subset of $\mathbb{C}$, which is not all of $\mathbb{C}$. Then there exists a bijective holomorphism $f : U \to \{z \in \mathbb{C} : |z| < 1\}$.*

**Remark 11.6.** This means $f$ is conformal as well. [xxx talk about $f'(z) \neq 0$.]

An important topic on qualifying exams is the construction of a conformal map sending one region to another, say, the upper half-plane to the unit disk. The key to success here is the **big five**:

- $z \mapsto e^z$
- $z \mapsto \sin(z)$

- $z \mapsto 1/z$.
- $z \mapsto z + 1/z$
- $z \mapsto \frac{z-i}{z+i}$

Also there are the more obvious **little three**:

- $z \mapsto iz$
- $z \mapsto z + c$
- Composition.

Of course, a qualifying-exam author may devise something nefarious, unaddressable by items on this list. However, one hopes that some ad-hoc reasoning of the form given below might suffice to get a person through the morning.

It turns out to be most important to tabulate what each of these does to

- the upper half-plane,
- the unit disk,
- horizontal strips, and
- vertical strips.

As much as I'd love to doodle up some nice figures (and someday I may), you can find these in any complex-analysis text, Mathworld, etc. My added value here is to focus on which conformal maps come up most often on the quals. Here are some useful facts (letting $z = x + iy$ throughout):

- $e^z$ takes the horizontal strip $0 \leq y \leq \pi$ to the upper half-plane. To remember this, write out $e^z = e^{x+iy} = e^x e^{iy}$. Then think about what happens when $x$ or $y$ is held fixed. For each $x$, we have a semicircle in the upper half-plane with radius $e^x$.
- $e^z$ takes the half-horizontal strip $0 \leq y \leq \pi$ and $x > 0$ to the upper half-plane, minus the unit disk. Since $x > 0$, the radii of the circles are $e^x > 1$.
- $e^z$ takes the half-horizontal strip $0 \leq y \leq \pi$ and $x \leq 0$ to the half of the unit disk in the upper half-plane: Since $x \leq 0$, the radii of the circles are $e^x > 1$.
- $e^z$ takes the vertical strip $0 \leq x \leq 1$ to the annulus with inner radius 1 and outer radius $e$: For each $x$ from 0 to 1, $e^x$ ranges from 1 to $e$ and so $e^x e^{iy}$ traces out circles as $y$ runs along. Similarly, we can find the image of a rectangle under $e^z$.
- $\sin z$ takes vertical lines to ellipses and horizontal lines to hyperbols. (Be sure to find a picture of this map — it's very pretty.) To remember this, recall that

  $$\sin(x + iy) = \sin x \cosh y + i \cos x \sinh y.$$

  The images of horizontal line segments are the upper halves of ellipses with horizontal radius $\cosh y$ and vertical radius $\sinh y$; the images of vertical lines are hyperbolas. (See my geometry-topology qual solutions, Fall 2004 #2, for a derivation of the formula, and further discussion of this map.)

- $1/z$ sends the unit circle to itself, and sends the interior of the unit circle to its exterior and vice versa.
- $1/z$ sends the circle centered at 1 (so the left edge is at $x = 0$ and the right edge is at $x = 2$) to the half-plane $x \geq 1/2$. And vice versa of course, since $1/(1/z) = 1$.
- $1/z$ ....
- $1/z$ ....
- $1/z$ ....
- $z + 1/z$ sends .... Note CB p. 27: $u = (r + 1/r)\cos\theta$; $v = (r - 1/r)\sin\theta$ (ellipse!). Spell these out and from that derive a mnemonic for the image of the upper and lower half-disks.
- $(z-1)/(z+1)$ sends .... Also mention $(z-i)/(z+i)$ in terms of pre-rotation. Also xref forward to LFTs.

**Example 11.7.** $\triangleright$ Let $z = x + iy$. Find a conformal map sending the horizontal strip $0 \leq y \leq 1$ to the unit disk.

Reading off the above information, we see $e^z$ sending the horizontal strip $0 \leq y \leq \pi$ to the upper half-plane, and $(z-1)/(z+1)$ sending the right half-plane to the unit disk. So, compose:

- Map the horizontal strip $0 \leq y \leq 1$ to the horizontal strip $0 \leq y \leq \pi$ by multiplication by $\pi$.
- Send that to the right half-plane by $e^z$.
- Rotate that to the upper half-plane by multiplying by $i$.
- Send that to the unit disk by $(z-1)/(z+1)$.

Let's write out the compositions one step at a time. (I know I'll make an algebra mistake if I try to write it all out at once.)

$$\begin{aligned} w_1 &= \pi z \\ w_2 &= e^{\pi z} \\ w_3 &= ie^{\pi z} \\ w_4 &= \frac{ie^{\pi z} - 1}{ie^{\pi z} + 1}. \end{aligned}$$

$\triangleleft$

xxx double-check this.

xxx do it using $1/z$.

# 12 Linear fractional transformations

(This section is due mostly to Victor Piercey.)

**Definition 12.1.** A **linear fractional transformation** (also known as a **Möbius transformation**) is a map $T : \mathbb{C} \to \mathbb{C}$ given by

$$T(z) = \frac{az + b}{cz + d}$$

where $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$.

**Definition 12.2.** I call the quantity $ad - bc$, for lack of a better name, the **determinant** of $T$.

Next, several facts.

**Proposition 12.3.** *LFTs are invertible via*

$$T^{-1}(z) = \frac{-dz + b}{cz - a}.$$

*Proof.* Compose both directions and simplify, obtaining the identity each time. □

**Proposition 12.4.** *The composition of LFTs is another LFT.*

*Proof.* Write out the composition of

$$z \mapsto \frac{az + b}{cz + d} \qquad \text{and} \qquad z \mapsto \frac{ez + f}{gz + h}.$$

The determinant (after some algebra) factors as

$$(ad - bc)(eh - fg),$$

i.e. the product of determinants of the two original LFTs, which is non-zero iff both the original determinants are non-zero. □

Since the identity is an LFT and composition is associative, we have shown:

**Proposition 12.5.** *The set of all LFTs is a group with respect to composition.*

**Remark 12.6.** LFTs extend to $\hat{\mathbb{C}}$ with

$$T(\infty) = a/c \qquad \text{and} \qquad T(-d/c) = \infty.$$

(Recall that $a/c = \infty$ if $c = 0$: see example 4.4.) From here on out, we will consider our LFTs to be on $\hat{\mathbb{C}}$.

**Remark 12.7.** The group of all LFTs is sometimes called $\mathrm{Aut}(\hat{\mathbb{C}})$, isomorphic to $\mathrm{PGL}_2(\mathbb{C})$; it is the automorphism group of the Riemann sphere. (See the very nice Wikipedia article on Linear Fractional Transformations.) I'm not aware of any (as-yet-written) qual problems which make use of this fact, but keep an eye out for covering-space questions.

**Lemma 12.8.** *Every LFT other than the identity has at most two fixed points.*

*Proof.* Suppose we have a fixed point, namely,

$$z = T(z) = \frac{az + b}{cz + d}.$$

Clearing denominators gives

$$cz^2 + (d - a)z - b = 0.$$

- If $c \neq 0$, this is a quadratic with complex coefficients, so either there are two distinct roots, or there is one repeated root.
- If $c = 0$ and $d \neq a$ then it is linear, with one root; it fixes 0 and $\infty$.

- If $c = 0$ and $d = a$ and $b \neq 0$ then it has no roots. However $T$ does fix $\infty$.
- If $c = 0$ and $d = a$ and $b = 0$ then it has infinitely many roots, but then one of the following is true: either $d = a = 0$, in which case $a = b = c = d = 0$ and $T$ is not an LFT, or $a, d \neq 0$ and $T$ is the identity transformation.

□

This definition is used in the proof of the following lemma.

**Definition 12.9.** Given three distinct complex numbers $z_1$, $z_2$, and $z_3$, let $A(z)$ be the LFT given by

$$A(z) = \left(\frac{z - z_1}{z - z_3}\right)\left(\frac{z_2 - z_3}{z_2 - z_1}\right).$$

This is called the **implicit form** for $A$.

**Proposition 12.10.** *These are, in fact, LFTs.*

*Proof.* Distributing the top and bottom, we get

$$\left(\frac{z - z_1}{z - z_3}\right)\left(\frac{z_2 - z_3}{z_2 - z_1}\right) = \frac{(z_2 - z_3)z - (z_2 - z_3)z_1}{(z_2 - z_1)z - (z_2 - z_1)z_3}.$$

This has determinant equal to

$$(z_2 - z_3)(z_2 - z_1)(z_3 - z_1).$$

This is non-zero iff $z_1$, $z_2$, and $z_3$ are distinct. □

The next result is of key importance for qualifying exams.

**Proposition 12.11.** *A linear fractional transformation is uniquely specified by its action on three distinct points.*

*Proof, from Fisher's text.* Let the three distinct points be $z_1$, $z_2$, and $z_3$, with images $w_1 = T(z_1)$, $w_2 = T(z_2)$, and $w_3 = T(z_3)$. Define one LFT, $A$, such that

$$A(z_1) = 0, \qquad A(z_2) = 1, \qquad \text{and} \qquad A(z_3) = \infty;$$

define another LFT, $B$, such that

$$B(w_1) = 0, \qquad B(w_2) = 1, \qquad \text{and} \qquad B(w_3) = \infty.$$

The existence of $A$ and $B$ is guaranteed by the existence of their implicit forms. Then

$$B^{-1}Az_i = w_i$$

for $i = 1, 2, 3$, so

$$T^{-1}B^{-1}Az_i = z_i.$$

for $i = 1, 2, 3$. This is a composition of LFTs, hence an LFT, with three fixed points. By the lemma, it is the identity transformation, i.e. $T = B^{-1}A$. □

The following facts aid in the construction of LFTs.

**Proposition 12.12.** *LFTs map circles and lines to circles or lines.*

**Remark 12.13.** LFTs are compositions of

$$z \mapsto az, \qquad z \mapsto z+b, \qquad \text{and} \qquad z \mapsto 1/z.$$

**Technique 12.14** (Churchill and Brown)**.** To write down an LFT given three points $z_1$, $z_2$, and $z_3$, and their images $w_1$, $w_2$, and $w_3$, as long as none of these points is at infinity, write down

$$\left(\frac{w-w_1}{w-w_3}\right)\left(\frac{w_2-w_3}{w_2-w_1}\right) = \left(\frac{z-z_1}{z-z_3}\right)\left(\frac{z_2-z_3}{z_2-z_1}\right)$$

and solve for $z$. If any $z_i$ or $w_i$ is $\infty$, replace it with $1/z_i$ or $1/w_i$, respectively, then set it to zero after simplifying.

**Example 12.15.** $\triangleright$ Find an LFT sending 0 to 1, 1 to $\infty$, and $\infty$ to 0.

We have

$$\begin{aligned} z_1 &= 0 & w_1 &= 1 \\ z_2 &= 1 & w_2 &= \infty \\ z_3 &= \infty & w_3 &= 0. \end{aligned}$$

Then we need to solve for $w$:

$$\begin{aligned} \left(\frac{w-1}{w}\right)\left(\frac{1/w_2}{1/w_2-1}\right) &= \left(\frac{z}{z-1/z_3}\right)\left(\frac{1-1/z_3}{1}\right) \\ \left(\frac{w-1}{w}\right)\left(\frac{1}{1-w_2}\right) &= \left(\frac{z}{z_3z-1}\right)\left(\frac{z_3-1}{1}\right) \\ \left(\frac{w-1}{w}\right) &= z \\ w-1 &= zw \\ w-zw &= 1 \\ w &= \frac{1}{1-z}. \end{aligned}$$

Now check, using definition 4.3:

$$\begin{aligned} \frac{1}{1-0} &= 1 \\ \frac{1}{1-1} &= \infty \\ \frac{1}{1-\infty} &= \lim_{z\to 0}\frac{1}{1-1/z} \\ &= \lim_{z\to 0}\frac{z}{z-1} = 0 \end{aligned}$$

as desired. $\triangleleft$

Alternatively, one can plug in each of the three inputs and solve the resulting system of equations.

**Example 12.16.** $\triangleright$ Use the same values as in the previous example.

$$\begin{aligned}
\frac{0a+b}{0c+d} &= 1 \\
\frac{1a+b}{1c+d} &= \infty \\
\frac{a+0b}{c+0d} &= 0 \\
\\
\frac{b}{d} &= 1 \\
\frac{c+d}{a+b} &= 0 \\
\frac{a}{c} &= 0.
\end{aligned}$$

The third equation gives $a = 0$; the first gives $b = d$; then, the second becomes $(c + b)/b = 0$ from which $c = -b$. It might seem odd that we are solving three equations for four unknowns, but remember that a constant multiple on the top and bottom cancels out, so there are really only three degrees of freedom. Take $b = 1$. Then $d = b = 1$, $c = -b = -1$, and $a = 0$ give

$$z \mapsto \frac{1}{1-z}$$

as before. $\triangleleft$

# References

[**CB**] Churchill, R.V. and Brown, J.W. *Complex Variables and Applications* (5th ed.). McGraw-Hill, 1990.

[**Wik**] Articles on *holomorphic* and *non-analytic smooth function*. http://en.wikipedia.org.

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