Problems in stochastic differential equations

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Abstract

The following are some problem solutions from a course in stochastic differential equations I took from Jan Wehr in spring 2008 at the University of Arizona. Here I collect various topics related to the Itō formula: statement of the formula, special cases, integration techniques, and solutions to some interesting SDEs. For the latter, I include analytical results as well as plots obtained from numerical simulations.

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1 Reference

Before presenting problem solutions, I will first tabulate some results for ready reference. The following are collected from [**Øks**] and from my problem solutions, using definitions and notation as in [**Øks**]. See also [**GS**] and/or [**Law**] for more background information on Brownian motion, martingales, and stochastic differential equations.

1.1 The Itō formula in one dimension

Proposition 1.1. *If*

$$dX_t = e(X_t, t)\, dt + f(X_t, t)\, db_t \tag{1.1}$$

is an Itō process and

$$y = g(t, x)$$

is a C^2 map from $[0, +\infty) \times \mathbb{R}$ to $\mathbb{R}$, then

$$Y_t = g(t, X_t)$$

is again an Itō process with

$$dY_t = \frac{\partial g}{\partial t}(t, X_t)\, dt + \frac{\partial g}{\partial x}(t, X_t)\, dX_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t, X_t)\, dX_t^2. \tag{1.2}$$

Remark. Recall that we use the Itō calculus rules

$$db_t^2 = dt, \qquad dt^2 = 0, \qquad dt\, db_t = 0$$

when expanding dX_t^2.

Remark. If we expand dX_t (equation 1.1) into 1.2, we obtain an alternate expression

$$dX_t^2 = e^2\, dt^2 + 2ef\, dt\, db_t + f^2 db_t^2 = f^2 dt \tag{1.3}$$

$$dY_t = \frac{\partial g}{\partial t}(t, X_t)\, dt + \frac{\partial g}{\partial x}(t, X_t)\, (e\, dt + f\, db_t) + \frac{f^2}{2}\frac{\partial^2 g}{\partial x^2}(t, X_t)\, dt \tag{1.4}$$

$$= \left(\frac{\partial g}{\partial t}(t, X_t) + e\,\frac{\partial g}{\partial x}(t, X_t) + \frac{f^2}{2}\frac{\partial^2 g}{\partial x^2}(t, X_t)\right)\, dt + f\,\frac{\partial g}{\partial x}(t, X_t)\, db_t. \tag{1.5}$$

The following two cases appear more often in the problem solutions than the general statement of proposition 1.1, so they are worth writing down to relieve the brain of the task of error-prone on-the-fly relabeling.

Proposition 1.2. *Special case with $e = 0$ and $f = 1$: If $x = g(t, u)$ is C^2 then*

$$X_t = g(t, b_t)$$

is an Itō process with

$$dX_t = \frac{\partial g}{\partial t}(t, b_t)\, dt + \frac{\partial g}{\partial u}(t, b_t)\, dX_t + \frac{1}{2}\frac{\partial^2 g}{\partial u^2}(t, b_t)\, dX_t^2 \tag{1.6}$$

$$= \left(\frac{\partial g}{\partial t}(t, b_t) + \frac{1}{2}\frac{\partial^2 g}{\partial u^2}(t, b_t)\right)\, dt + \frac{\partial g}{\partial u}(t, b_t)\, db_t. \tag{1.7}$$

Proposition 1.3. *Special case with $e = 0$ and $f = 1$ and no explicit time dependence: If*

$$X_t = g(b_t)$$

then

$$dX_t = \frac{1}{2}\frac{\partial^2 g}{\partial u^2}(b_t)\, dt + \frac{\partial g}{\partial u}(b_t)\, db_t. \tag{1.8}$$

1.2 The Itō formula in higher dimensions

Proposition 1.4. *Let*

$$\begin{pmatrix} dX_t^{(1)} \\ \vdots \\ dX_t^{(n)} \end{pmatrix} = \begin{pmatrix} u_1\,dt \\ \vdots \\ u_n\,dt \end{pmatrix} + \begin{pmatrix} v_{1,1} & \cdots & v_{1,m} \\ \vdots & & \vdots \\ v_{n,1} & \cdots & v_{n,m} \end{pmatrix} \begin{pmatrix} db_t^{(1)} \\ \vdots \\ db_t^{(m)} \end{pmatrix},$$

i.e. in matrix notation

$$d\mathbf{X}_t = \mathbf{u}\,dt + \mathbf{V}\,d\mathbf{b}_t.$$

(This is called am n-dimensional Itō process.) Let $\mathbf{g} : [0, +\infty) \times \mathbb{R}^n \to \mathbb{R}^p$ be a C^2 map:

$$\begin{pmatrix} y_1 \\ \vdots \\ y_p \end{pmatrix} = \begin{pmatrix} g_1(t, \mathbf{x}) \\ \vdots \\ g_p(t, \mathbf{x}) \end{pmatrix} = \begin{pmatrix} g_1(t, x_1, \ldots, x_n) \\ \vdots \\ g_p(t, x_1, \ldots, x_n) \end{pmatrix}.$$

Then

$$\mathbf{Y}_t = \mathbf{g}(t, \mathbf{X}_t)$$

is again an Itō process with kth component (for $k = 1, \ldots, p$)

$$dY_t^{(k)} = \frac{\partial g}{\partial t}(t, \mathbf{X}_t)\,dt + \sum_{i=1}^{n} \frac{\partial g_k}{\partial x_i}(t, \mathbf{X}_t)\,dX_t^{(i)} + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 g_k}{\partial x_i \partial x_j}(t, \mathbf{X}_t)\,dX_t^{(i)}\,dX_t^{(j)}.$$

Remark. Recall that we use the Itō calculus rules

$$db_t^{(i)}\,db_t^{(j)} = \delta_{ij}, \qquad dt^2 = 0, \qquad dt\,db_t^{(i)} = 0$$

when expanding $dX_t^{(i)}\,dX_t^{(j)}$.

Again, there is a special case when the input Itō process is simply n-dimensional Brownian motion; again, it is worth writing down as such since this case arises often.

Proposition 1.5. *Let $\mathbf{g} : \mathbb{R}^+ \times \mathbb{R}^m \to \mathbb{R}^n$ be a C^2 map:*

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} g_1(t, \mathbf{u}) \\ \vdots \\ g_n(t, \mathbf{u}) \end{pmatrix} = \begin{pmatrix} g_1(t, u_1, \ldots, u_m) \\ \vdots \\ g_n(t, u_1, \ldots, u_m) \end{pmatrix}.$$

Then

$$\mathbf{X}_t = \mathbf{g}(t, \mathbf{b}_t)$$

is again an Itō process with kth component (for $k = 1, \ldots, n$)

$$dX_t^{(k)} = \frac{\partial g}{\partial t}(t, \mathbf{b}_t)\,dt + \sum_{i=1}^{m} \frac{\partial g_k}{\partial u_i}(t, \mathbf{b}_t)\,db_t^{(i)} + \frac{1}{2} \sum_{i,j=1}^{m} \frac{\partial^2 g_k}{\partial u_i \partial u_j}(t, \mathbf{b}_t)\,db_t^{(i)}\,db_t^{(j)}.$$

1.3 The Itō formula from two dimensions to one

Proposition 1.6. *Let* $\mathbf{g} : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{R}$ *be a* C^2 *map*

$$z = g(t, x, y).$$

Then

$$Z_t = g(t, X_t, Y_t)$$

is again an Itō process with

$$\begin{aligned} dZ_t = & \frac{\partial g}{\partial t}(t, X_t, Y_t)\, dt + \frac{\partial g}{\partial x}(t, X_t, Y_t)\, dX_t + \frac{\partial g}{\partial y}(t, X_t, Y_t)\, dY_t \\ & + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t, X_t, Y_t)\, dX_t^2 + \frac{1}{2}\frac{\partial^2 g}{\partial y^2}(t, X_t, Y_t)\, dY_t^2 + \frac{\partial^2 g}{\partial x \partial y}(t, X_t, Y_t)\, dX_t\, dY_t. \end{aligned}$$

1.4 A derivation of the one-dimensional Itō formula

This is not a proof (see e.g. [**Øks**] section 4.1) — rather, it is a heuristic discussion intended to remind me why which terms appear where.

- Start with

$$dX_t = e(X_t, t)\, dt + f(X_t, t)\, db_t.$$

- Consider

$$Y_t = g(X_t, t).$$

- We need to show (using the formulation of equation 1.3)

$$dY_t = \left(\frac{\partial g}{\partial t}(t, X_t) + e\, \frac{\partial g}{\partial x}(t, X_t) + \frac{f^2}{2}\frac{\partial^2 g}{\partial x^2}(t, X_t) \right) dt + f\, \frac{\partial g}{\partial x}(t, X_t)\, db_t.$$

- Using the shorthand partial-derivative notation $g_t = \partial g / \partial t$ etc., Taylor-expand dY as

$$\begin{aligned} dY &= g_t\, dt + g_x\, dX + \frac{g_{tt}}{2}\, dt^2 + \frac{g_{xx}}{2}\, dX^2 + g_{tx}\, dt\, dX + \text{higher order terms} \\ &= g_t\, dt + g_x(e\, dt + f\, db_t) + \frac{g_{tt}}{2}\, dt^2 + \frac{g_{xx}}{2}(e\, dt + f\, db_t)^2 + g_{tx}\, dt(e\, dt + f\, db_t) + \ldots. \end{aligned}$$

- Use $dt^2 = 0$, $dt\, db_t = 0$, $db_t^2 = dt$:

$$\begin{aligned} dY &= g_t\, dt + g_x(e\, dt + f\, db_t) + \frac{g_{xx}}{2} f^2\, dt \\ &= g_t\, dt + e\, g_x\, dt + f^2\, \frac{g_{xx}}{2}\, dt + f\, g_x\, db_t \\ &= \left(g_t + e\, g_x + f^2\, \frac{g_{xx}}{2} \right) dt + f\, g_x\, db_t. \end{aligned}$$

1.5 The stochastic product rule

Proposition 1.7. *Let X_t, Y_t be Itō processes. Then*

$$d(X_t\,Y_t) = X_t\,dY_t + Y_t\,dX_t + dX_t\,dY_t.$$

Proof. Using the Itō formula in the form of proposition 1.6 with $g(x,y) = xy$, we have

$$\frac{\partial g}{\partial t} = 0, \qquad \frac{\partial g}{\partial x} = y, \qquad \frac{\partial g}{\partial y} = x, \qquad \frac{\partial^2 g}{\partial x^2} = 0, \qquad \frac{\partial^2 g}{\partial y^2} = 0, \qquad \text{and} \qquad \frac{\partial^2 g}{\partial x \partial y} = 1.$$

Then

$$d(X_t\,Y_t) = Y_t\,dX_t + X_t\,dY_t + dX_t\,dY_t.$$

□

1.6 Stochastic integration by parts

Proposition 1.8. *For Itō processes X_t and Y_t,*

$$\int_0^t X_s dY_s = X_t Y_t - X_0 Y_0 - \int_0^t Y_s\,dX_s - \int_0^t dX_s\,dY_s.$$

Proof. Using the stochastic product rule (proposition 1.7), we have

$$\begin{aligned} d(X_t\,Y_t) &= X_t\,dY_t + Y_t\,dX_t + dX_t\,dY_t \\ X_t\,dY_t &= d(X_t\,Y_t) - Y_t\,dX_t - dX_t\,dY_t. \end{aligned}$$

Integrating from 0 to t gives

$$\begin{aligned} \int_0^t X_s\,dY_s &= \int_0^t d(X_s\,Y_s) - \int_0^t Y_s\,dX_s - \int_0^t dX_s\,dY_s \\ &= X_t Y_t - X_0 Y_0 - \int_0^t Y_s\,dX_s - \int_0^t dX_s\,dY_s. \end{aligned}$$

□

Corollary 1.9. *For f_t not depending on b_t, we have*

$$\int_0^t f_s\,db_s = f_t b_t - \int_0^t b_s\,df_s.$$

Proof. Putting $X_t = f_t$ and $Y_t = b_t$, the proposition gives

$$\int_0^t f_s\,db_s = f_t b_t - \int_0^t b_s\,df_s - \int_0^t df_s\,db_s$$

since $b_0 = 0$. By the Itō calculus rule $dt\,db_t = 0$, the last integral vanishes as long as there is no db_t in df_t. □

1.7 The Itō isometry

Proposition 1.10. *For all f such that the Itō integral is defined,*

$$E\left[\left(\int_{t_1}^{t_2} f_s\, db_s\right)^2\right] = E\left[\int_{t_1}^{t_2} f_s^2\, ds\right].$$

1.8 Expectation of integrals

Proposition 1.11. *For all f such that the Itō integral is defined,*

$$E\left[\int_{t_1}^{t_2} f_s db_s\right] = 0.$$

Proof. This is property (iii) on p. 30 of Øksendahl. The proof is quick for step functions, since

$$E[c\, b_t] = cE[\, b_t] = 0$$

on each piece; for the general case, one takes limits of step functions. □

2 Damped Brownian motion (problem 1.5a)

Prove that

$$X_t = \frac{b_t}{1+t}$$

solves

$$dX_t = \frac{-1}{1+t}\,X_t\,dt + \frac{1}{1+t}\,db_t.$$

Proof. Use the Itō formula in the form of proposition 1.2 with

$$g(t,x) = \frac{x}{1+t}.$$

We have

$$\frac{\partial g}{\partial t} = \frac{-x}{(1+t)^2}, \qquad \frac{\partial g}{\partial x} = \frac{1}{1+t}, \qquad \text{and} \qquad \frac{\partial^2 g}{\partial x^2} = 0.$$

Then

$$\begin{aligned} dX_t &= \frac{-b_t}{(1+t)^2}\,dt + \frac{db_t}{1+t} \\ &= \frac{-1}{1+t}\,X_t\,dt + \frac{1}{1+t}\,db_t. \end{aligned}$$

□

Figure 1 shows five realizations of X_t with t from 0 to 10 in steps of 0.005. The exact solutions and approximate solutions (using naive a first-order Euler method) are both plotted, but are indistinguishable at this scale.

Python source code for all solutions plotted in this document may be found in the directory

`http://math.arizona.edu/~kerl/doc/565C/code/`

I will not include source code for each problem. Here, however, is the Python code for this problem: this gives the flavor of the numerical approach.

```
import random        # For random.gauss: generation of normal deviates
from math import *   # For sqrt
T  = 10.0            # End time
nt = 2000            # Number of mesh points
dt = T/nt            # Time step
t  = 0.0             # Initial time
Bt = 0.0             # Initial value of Brownian motion
Xexact  = Bt / (1+t) # Initial exact solution
Xapprox = Bt / (1+t) # Initial approximate solution
while (t <= T):
    Xexact = Bt / (1+t)
    print "%11.7f %11.7f %11.7f %9.3e" % (t, Xexact, Xapprox, Xexact-Xapprox)
    dB = random.gauss(0, sqrt(dt))
    # First-order Euler update:
    Xapprox += -1/(1+t) * Xapprox * dt + 1/(1+t) * dB
    Bt += dB; t += dt
```

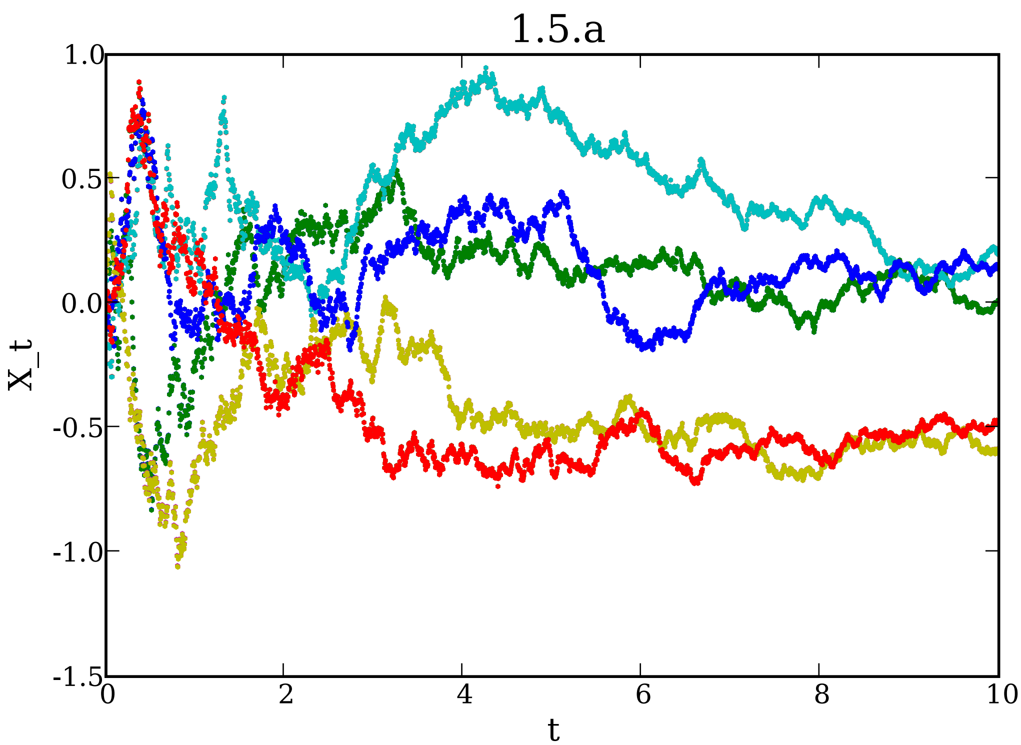


Figure 1: Problem 1.5.a: $X_t = b_t/(1+t)$.

3 Brownian motion on an ellipse (problem 1.5b)

Let

$$X_t = A\cos(b_t)$$
$$Y_t = B\sin(b_t),$$

where $A, B > 0$. *Show that these solve*

$$dX_t = -\frac{1}{2}X_t\,dt - \frac{A}{B}Y_t\,db_t$$
$$dY_t = -\frac{1}{2}Y_t\,dt + \frac{B}{A}X_t\,db_t.$$

Proof. Use the Itō formula (once for X_t and once for Y_t) in the form of proposition 1.5 with $m = 1$, $n = 2$, and

$$g(t, x) = A\cos(x)$$
$$h(t, x) = B\sin(x).$$

□

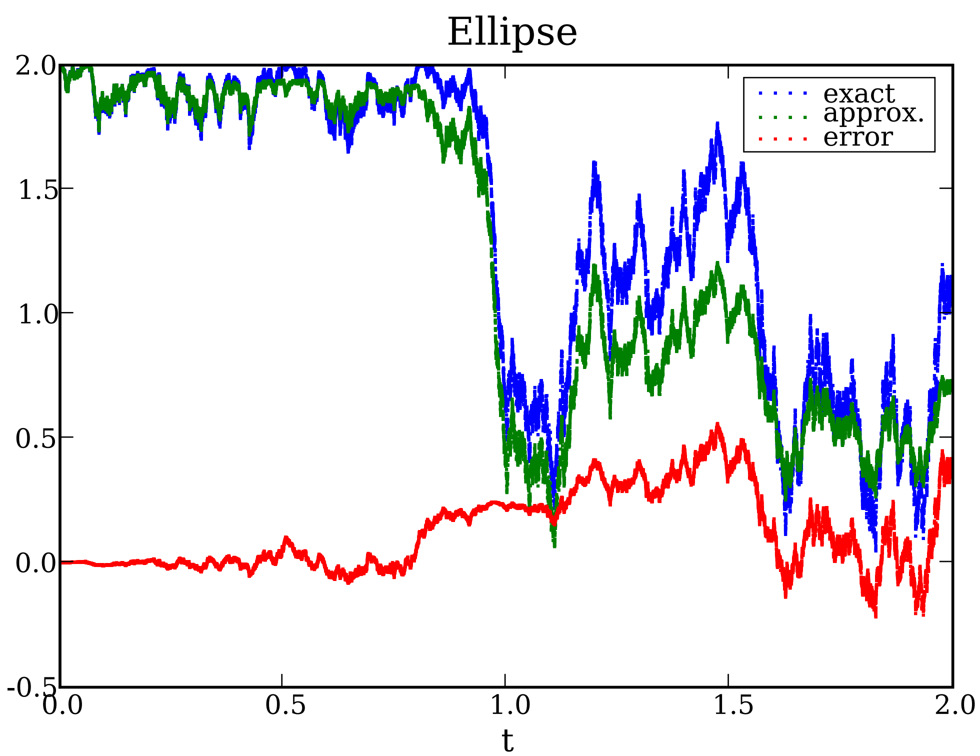


Figure 2: Problem 1.5.b: Cosine part of Brownian motion on an ellipse with $A = 2$ and $B = 3$.

We have

$$\frac{\partial g}{\partial t} = 0, \qquad \frac{\partial h}{\partial t} = 0,$$
$$\frac{\partial g}{\partial x} = -A\sin(x), \qquad \frac{\partial h}{\partial x} = B\cos(x),$$
$$\frac{\partial^2 g}{\partial x^2} = -A\cos(x), \qquad \frac{\partial^2 h}{\partial x^2} = -B\sin(x).$$

Then

$$dX_t = -\frac{A}{2}\cos(b_t)\,dt - A\sin(b_t)\,db_t$$
$$dY_t = -\frac{B}{2}\sin(b_t)\,dt + B\cos(b_t)\,db_t$$

$$dX_t = -\frac{1}{2}X_t\,dt - \frac{A}{B}\,Y_t\,db_t$$
$$dY_t = -\frac{1}{2}Y_t\,dt + \frac{B}{A}\,X_t\,db_t.$$

Figure 2 shows one realization of X_t with t from 0 to 2 in steps of 0.0001, with $A = 2$ and $B = 3$. The approximate solution is again a naive first-order Euler; the error control is poor here. One may take this as an advertisement for higher-order numerical methods, and/or as an advertisement for the value of closed-form solutions.

4 Exponential models with noise (problem 1.5.c)

4.1 Exponential growth with noise (problem 1.5.c.i)

Solve the equation

$$dX_t = X_t\,dt + db_t$$

with an arbitrary constant initial condition $X_0 = c$.

We may try the integrating factor e^{-t} and compare with $d(e^{-t}X_t)$. The equation to be solved becomes

$$e^{-t}\,dX_t = e^{-t}X_t\,dt + e^{-t}\,db_t.$$

On the other hand, applying the stochastic product rule (1.7), we have

$$\begin{aligned} d(e^{-t}X_t) &= d(e^{-t})X_t + e^{-t}dX_t + d(e^{-t})dX_t \\ &= -e^{-t}X_t dt + e^{-t}(X_t\,dt + db_t) - e^{-t}\,dt(X_t\,dt + db_t) \\ &= -e^{-t}X_t\,dt + e^{-t}X_t\,dt + e^{-t}db_t - e^{-t}X_t dt^2 - e^{-t}dt\,db_t \\ &= e^{-t}\,db_t. \end{aligned}$$

Integrating both sides, we have

$$\begin{aligned} \int_0^t d(e^{-s}X_s) &= \int_0^t e^{-s}\,dbs \\ e^{-t}X_t - X_0 &= \int_0^t e^{-s}\,db_s \\ e^{-t}X_t &= c + \int_0^t e^{-s}\,db_s \\ X_t &= ce^t + e^t\int_0^t e^{-s}\,db_s. \end{aligned}$$

Figure 3 shows five realizations of X_t with t from 0 to 2 in steps of 0.001. Here, the exact and approximate solutions are indistinguishable. The expected value is also shown. This is

$$\begin{aligned} E[X_t] &= E[ce^t] + E\left[e^t\int_0^t e^{-s}\,db_s\right] \\ &= ce^t \end{aligned}$$

using the non-stochasticity of the first term and proposition 1.11 for the second term.

4.2 Exponential decay with decaying noise (problem 1.5.c.ii)

Solve the equation

$$dX_t = -X_t\,dt + e^{-t}db_t$$

with an arbitrary constant initial condition $X_0 = c$.

We may try the integrating factor e^t and compare with $d(e^tX_t)$. The equation to be solved becomes

$$e^t\,dX_t = -e^tX_t\,dt + db_t.$$

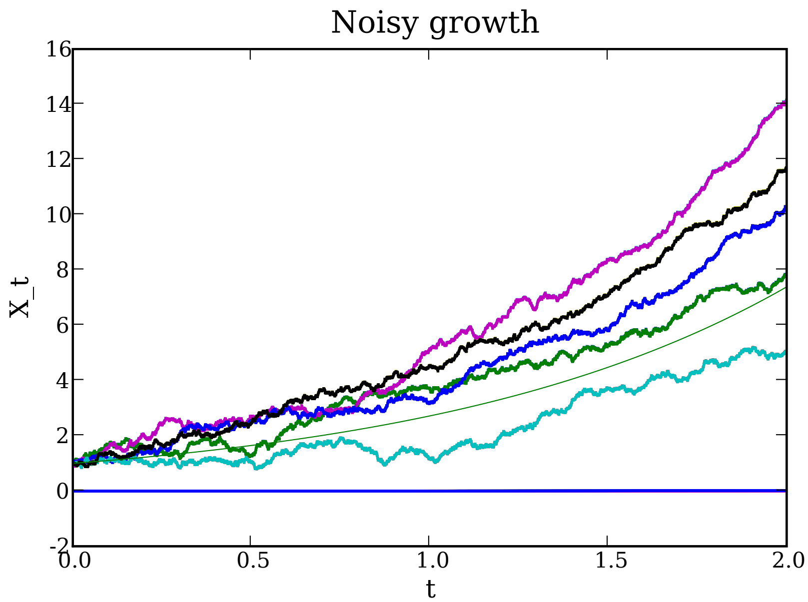


Figure 3: Problem 1.5.c.i: $dX_t = X_t\, dt + db_t$, along with $E[X_t]$.

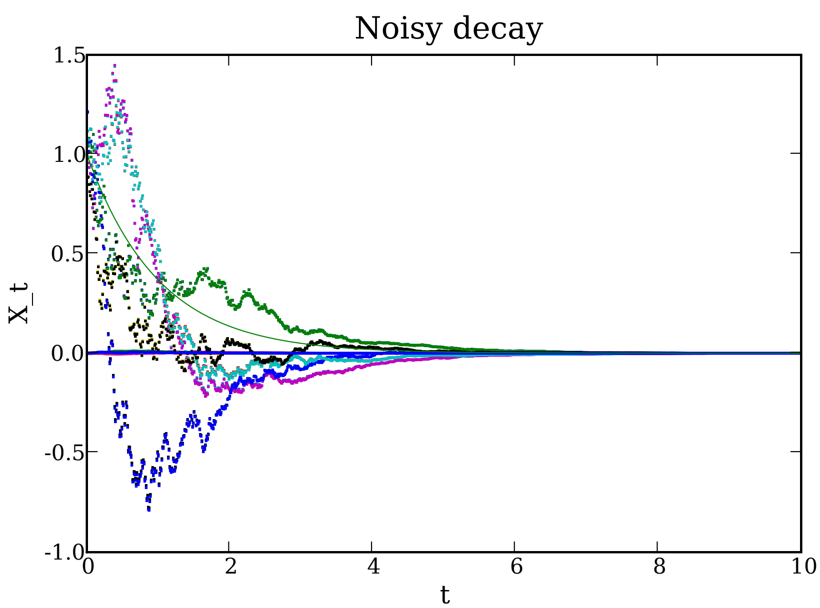


Figure 4: Problem 1.5.c.ii: $dX_t = -X_t\, dt + e^{-t} db_t$, along with $E[X_t]$.

Applying the stochastic product rule (1.7), we have

$$\begin{aligned} d(e^t X_t) &= e^t X_t dt + e^t\, dX_t + e^t\, dt\, dX_t \\ &= e^t X_t\, dt + e^t(-X_t\, dt + e^{-t}\, db_t) + e^t\, dt(-X_t\, dt + e^{-t}\, db_t) \\ &= e^t X_t\, dt - e^t X_t\, dt + db_t \\ &= db_t. \end{aligned}$$

Integrating both sides, we have

$$\begin{aligned} \int_0^t d(e^s X_s) &= \int_0^t dbs \\ e^t X_t &= X_0 = b_t - b_0 \\ X_t &= (c + b_t)e^{-t}. \end{aligned}$$

Figure 4 shows five realizations of X_t with t from 0 to 10 in steps of 0.001. Here, the exact and approximate solutions are indistinguishable. The expectation $E[X_t]$ is also shown. This is

$$E[X_t] = E[ce^{-t}] + E[b_t e^{-t}] = ce^{-t}$$

since b_t has mean 0 for all t.

5 Brownian bridges (problem 1.5.d)

Part (i): *Verify that the process*

$$Y_t = A(1-t) + Bt + (1-t)\int_0^t \frac{db_s}{1-s}, \qquad 0 \le t < 1,$$

is a solution of

$$dY_t = \frac{B - Y_t}{1-t}\,dt + db_t, \qquad 0 \le t < 1, Y_0 = A.$$

Differentiating the first equation, we have

$$\begin{aligned} dY_t &= Ad(1-t) + B\,dt + d\left((1-t)\int_0^t \frac{db_s}{1-s}\right) \\ &= (B-A)dt + d\left((1-t)\int_0^t \frac{db_s}{1-s}\right). \end{aligned}$$

Using the stochastic product rule requires taking d of the integral. I want to use the second fundamental theorem of calculus, but that applies to Lebesgue integrals; I don't know how to apply it to a stochastic integral. In order to convert from the latter to the former, I will use integration by parts (proposition 1.9), namely,

$$\int_0^t f_s\,db_s = f_t b_t - \int_0^t b_s\,df_s.$$

Here $f(s) = (1-s)^{-1}$ and $df_s = ds/(1-s)^2$. Thus

$$\int_0^t \frac{db_s}{1-s} = \frac{b_t}{1-t} - \int_0^t \frac{b_s\,ds}{(1-s)^2}.$$

Then

$$\begin{aligned} d\int_0^t \frac{db_s}{1-s} &= d\left(\frac{b_t}{1-t}\right) - d\int_0^t \frac{b_s\,ds}{(1-s)^2} \\ &= \frac{db_t}{1-t} + \frac{b_t\,dt}{(1-t)^2} + db_t\,dt - \frac{b_t\,dt}{(1-t)^2} \\ &= \frac{db_t}{1-t}. \end{aligned}$$

Returning to dY_t we have

$$\begin{aligned} dY_t &= (B-A)dt - dt\int_0^t \frac{db_s}{1-s} + (1-t)\,d\int_0^t \frac{db_s}{1-s} - dt\,d\int_0^t \frac{db_s}{1-s} \\ &= (B-A)dt - dt\int_0^t \frac{db_s}{1-s} + (1-t)\,\frac{db_t}{1-t} - dt\frac{db_t}{1-t} \\ &= (B-A)dt - dt\int_0^t \frac{db_s}{1-s} + db_t. \end{aligned}$$

I am asked to show

$$dY_t = \frac{B - Y_t}{1-t}\,dt + db_t.$$

I need to show

$$B - A - \int_0^t \frac{db_s}{1-s} = \frac{B - Y_t}{1-t}$$

which is true if

$$B(1-t) - A(1-t) - (1-t)\int_0^t \frac{db_s}{1-s} = B - Y_t$$

which is true if

$$B(1-t) - A(1-t) - (1-t)\int_0^t \frac{db_s}{1-s} = B - A(1-t) - Bt - (1-t)\int_0^t \frac{db_s}{1-s}$$

which is true if

$$B(1-t) - A(1-t) - (1-t)\int_0^t \frac{db_s}{1-s} = B - A(1-t) - Bt - (1-t)\int_0^t \frac{db_s}{1-s}$$

which is true.

Figure 5 shows twenty realizations of Y_t with t from 0 to 1 in steps of 0.001. The approximate solutions are plotted.

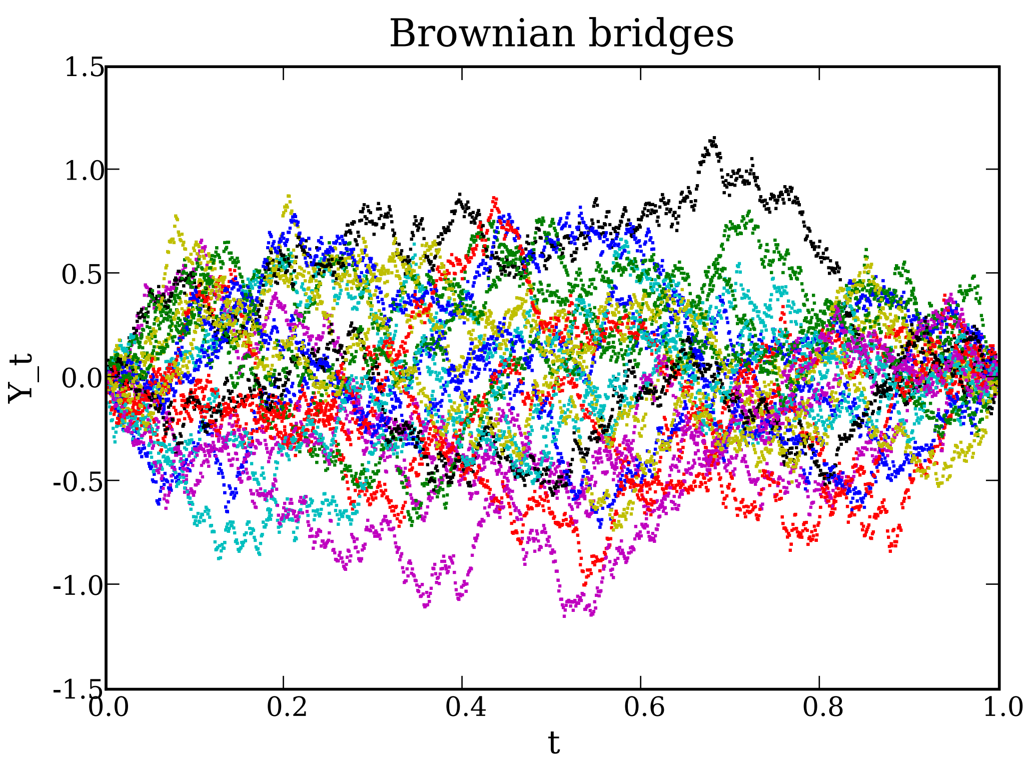


Figure 5: Problem 1.5.d: Brownian bridges running from zero to zero in time one.

Part (ii): *Show that* $\lim_{t\to 1} Y_t = B$ *with probability 1.*

Proof. Since

$$Y_t = A(1-t) + Bt + (1-t)\int_0^t \frac{db_s}{1-s},$$

We have

$$\lim_{t\to 1} Y_t = B + (1-t)\lim_{t\to 1}\int_0^t \frac{db_s}{1-s}.$$

The last term goes to zero as long as

$$\left|\int_0^t \frac{db_s}{1-s}\right| < \infty \quad w.p.1,$$

or at least if it goes to infinity slower than $1/(1-t)$.

Perhaps the Itō isometry (proposition 1.10) will help. With

$$f_s = \frac{1-t}{1-s},$$

we have

$$E\left[(1-t)^2\left(\int_0^t \frac{db_s}{1-s}\right)^2\right] = E\left[(1-t)^2\int_0^t \frac{ds}{(1-s)^2}\right].$$

But this is

$$E\left[(1-t)^2\int_0^t \frac{ds}{(1-s)^2}\right] = (1-t)^2\frac{t}{1-t} = t(1-t) < \infty$$

by elementary calculus.

□

Part (iii): *Find the mean and covariance of Y_t for $A = B = 0$.*

Here

$$Y_t = (1-t)\int_0^t \frac{db_s}{1-s}.$$

The variance is easily found — from part (ii), it is

$$E[Y_t^2] - E[Y_t]^2 = E[Y_t^2] = t(1-t).$$

The mean is zero by proposition 1.11.

For the covariance, assume without loss of generality that $s \le t$. Then — with the key being to separate the integral from 0 to t into an integral from 0 to s and another from s to t — we have

$$\begin{aligned}
E[Y_sY_t] &= E\left[(1-s)(1-t)\int_0^s \frac{db_u}{1-u}\int_0^t \frac{db_v}{1-v}\right] \\
&= (1-s)(1-t)\,E\left[\left(\int_0^s \frac{db_u}{1-u}\right)^2\right] + (1-s)(1-t)\,E\left[\int_0^s \frac{db_u}{1-u}\right]E\left[\int_s^t \frac{db_v}{1-v}\right].
\end{aligned}$$

The second expectation factors due to independence of the events; the first expecattion is as found in part (ii). Then we have

$$\begin{aligned}
E[Y_sY_t] &= (1-s)(1-t)\frac{s}{1-s} \\
&= s(1-t).
\end{aligned}$$

Note that for $s = t$ I recover the variance that I obtained using Itō isometry: this is reassuring.

Part (iv): *Verify that*

$$X_t := b_t - tb_1, \quad 0 \le t < 1,$$

has the same distribution as Y_t.

Proof. I will first compute the mean and covariance of X_t. These uniquely characterize Brownian motions; I would hope they also uniquely characterize Brownian bridges. (Regardless, the exercise is good practice for me.)

For the mean, we have

$$E[X_t] = E[b_t - tb_1] = 0 - t \cdot 0 = 0.$$

For the covariance, recalling that

$$E[b_s b_t] = s \wedge t$$

where $s \wedge t$ is the minimum of s and t, we have

$$\begin{aligned} E[X_s X_t] &= E[(b_s - sb_1)(b_t - tb_1)] \\ &= E[b_s b_t] - E[tb_1 b_s] - E[sb_1 b_t] + E[stb_1^2] \\ &= s \wedge t - t(s \wedge 1) - s(t \wedge 1) + st \\ &= s \wedge t - 2st + st \\ &= s \wedge t - st. \end{aligned}$$

Now, if $s \leq t$ then this is $s - st = s(1 - t)$, matching my result from part (iii). Happily, using the X_t formulation of the bridge I get the more elegant expression $s \wedge t - st$, which is valid whether $s \leq t$ or not — an expression which I might not have so easily gotten using the Y_t formulation.

Now, do these computations suffice to answer the original question? Do the zero-to-zero Brownian bridges form a Gaussian family? □

6 Complex Brownian motion (problem 1.5.e)

Let (a_t, b_t) *be two-dimensional Brownian motion. The complex-valued function*

$$c_t = a_t + ib_t$$

is called a complex Brownian motion. *Let* F *be an entire analytic function. For* $Z_t = F(c_t)$*, prove that*

$$dZ_t = F'(c_t)\, dc_t.$$

Proof. Write

$$z = F(w)$$

in the form

$$x + iy = g(u, v) + ih(u, v),$$

i.e. $w = u + iv$, $z = x + iy$, and $F = g + ih$. Then

$$\begin{aligned} X_t &= g(a_t, b_t) \\ Y_t &= h(a_t, b_t). \end{aligned}$$

Use the Itō formula in the form of proposition 1.5. Since $\partial g/\partial t$ and $\partial h/\partial t$ are zero, $da_t\, db_t$ is zero, and $da_t^2 = db_t^2 = dt$, we have

$$\begin{aligned} dX_t &= \frac{\partial g}{\partial u}\, da_t + \frac{\partial g}{\partial v}\, db_t + \frac{1}{2}\left(\frac{\partial^2 g}{\partial u^2} + \frac{\partial^2 g}{\partial v^2}\right) \\ dY_t &= \frac{\partial h}{\partial u}\, da_t + \frac{\partial h}{\partial v}\, db_t + \frac{1}{2}\left(\frac{\partial^2 h}{\partial u^2} + \frac{\partial^2 h}{\partial v^2}\right). \end{aligned}$$

Since F is entire analytic, its components g and h satisfy the Cauchy-Riemann equations

$$\frac{\partial g}{\partial x} = \frac{\partial h}{\partial y} \qquad \text{and} \qquad \frac{\partial h}{\partial x} = -\frac{\partial g}{\partial y}.$$

These in turn yield harmonicity of g and h:

$$\frac{\partial^2 g}{\partial u^2} + \frac{\partial^2 g}{\partial v^2} = \frac{\partial^2 h}{\partial u^2} + \frac{\partial^2 h}{\partial v^2} = 0.$$

Thus

$$\begin{aligned} dX_t &= \frac{\partial g}{\partial u}\, da_t + \frac{\partial g}{\partial v}\, db_t \\ dY_t &= \frac{\partial h}{\partial u}\, da_t + \frac{\partial h}{\partial v}\, db_t \end{aligned}$$

which is to say

$$dZ_t = F'(c_t)\, dc_t.$$

□

Figure 6 shows one realization of c_t with t from 0 to 5 in steps of 0.0005.

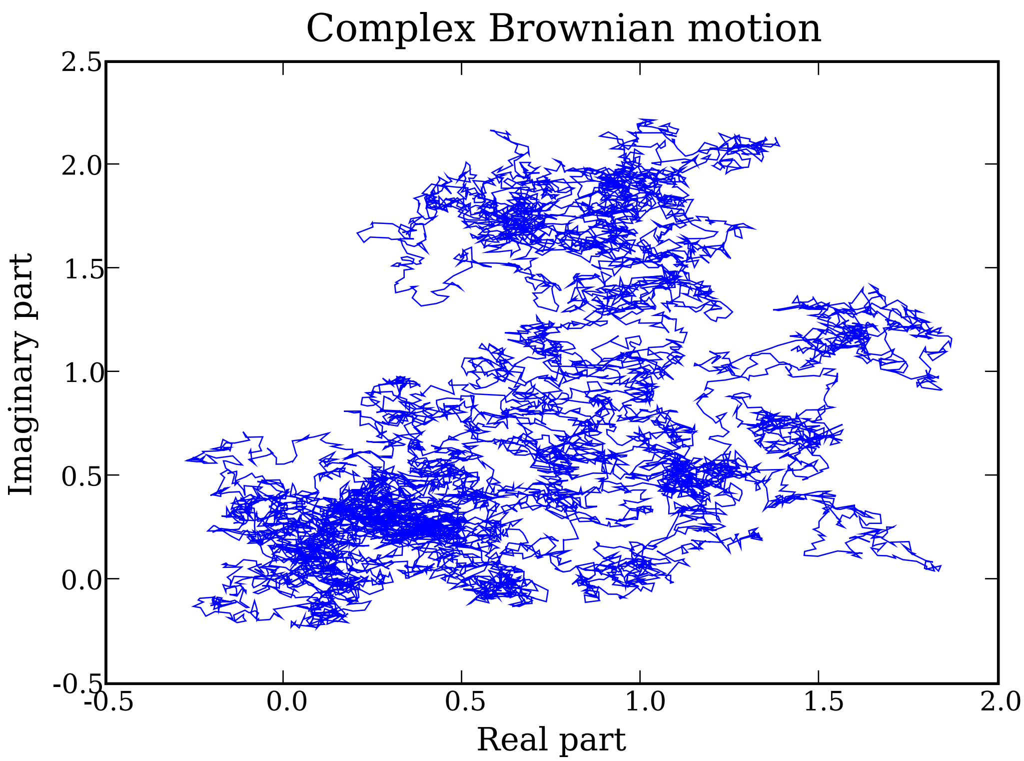


Figure 6: Problem 1.5.e: Complex Brownian motion.

7 Iterated stochastic integrals (problem 1.5.f)

The following is an important Itō formula for an iterated stochastic integral:

$$n! \int_0^t db_{u_n} \int_0^{u_n} db_{u_{n-1}} \cdots \int_0^{u_2} db_{u_1} = t^{n/2} h_n \left(\frac{b_t}{\sqrt{t}} \right),$$

where

$$h_n(x) = (-1)^n e^{\frac{t^2}{2}} \frac{d^n}{dx^n} \left(e^{\frac{-x^2}{2}} \right)$$

is the Hermite polynomial of degree n. Verify the formula for $n = 0, 1, 2, 3$.

Proof. First recall

$$\begin{aligned} h_0(x) &= 1, \\ h_1(x) &= x, \\ h_2(x) &= x^2 - 1, \\ h_3(x) &= x^3 - 3x. \end{aligned}$$

Case $n = 0$: This is immediate, as $0! = 1$.

Case $n = 1$: The left-hand side is

$$\int_0^t db_s = b_t;$$

the right-hand side is

$$t^{1/2}\, \frac{b_t}{\sqrt{t}} = b_t.$$

Case $n = 2$: The left-hand side is

$$\begin{aligned} 2\int_0^t db_s \int_0^s db_r &= 2\int_0^t db_s\, b_s \\ &= 2\left(\frac{1}{2}b_t^2 - \frac{t}{2}\right) \\ &= b_t^2 - t; \end{aligned}$$

the right-hand side is

$$t\left(t\left(\frac{b_t}{\sqrt{t}}\right)^2 - 1\right) = b_t^2 - t.$$

I take it that

$$\int_0^t db_s\, b_s = \frac{b_t^2 - t}{2}$$

is well-known. However, I will re-derive it, as it illustrates the technique used below. Use the Itō formula in the form of proposition 1.3. Guess

$$g(x) = \frac{x^2}{2},$$

so

$$\frac{\partial g}{\partial x} = x \qquad \text{and} \qquad \frac{\partial^2 g}{\partial x^2} = 1.$$

Then for

$$X_t = g(b_t)$$

we have

$$\begin{aligned} dX_t = d\left(\frac{b_t^2}{2}\right) &= \frac{1}{2}\frac{\partial^2 g}{\partial x^2}\, dt + \frac{\partial g}{\partial x}\, db_t \\ &= \frac{1}{2}\, dt + b_t\, db_t. \end{aligned}$$

Integrating from 0 to t we have

$$\begin{aligned} \int_0^t d\left(\frac{b_s^2}{2}\right) &= \frac{1}{2}\int_0^t ds + \int_0^t b_s\, db_s \\ \int_0^t b_s\, db_s &= \frac{b_t^2}{2} - \frac{t}{2}. \end{aligned}$$

Case $n = 3$: The left-hand side is

$$\begin{aligned} 6\int_0^t db_s \int_0^s db_r \int_0^r db_q &= 6\int_0^t db_s \int_0^s db_r\, b_r \\ &= 3\int_0^t db_s (b_s^2 - s) \\ &= 3\int_0^t b_s^2\, db_s - 3\int_0^t s\, db_s. \end{aligned}$$

The right-hand side is

$$\begin{aligned} t^{3/2} h_3\left(\frac{b_t}{\sqrt{t}}\right) &= t^{3/2}\left(\frac{b_t}{\sqrt{t}}\right)^3 - t^{3/2}\left(\frac{b_t}{\sqrt{t}}\right) \\ &= b_t^3 - t\, b_t. \end{aligned}$$

* * *

First we need to compute

$$\int_0^t b_s^2\, db_s.$$

Proceeding as above and guessing

$$g(x) = \frac{x^3}{3},$$

we have

$$\frac{\partial g}{\partial x} = x^2 \qquad \text{and} \qquad \frac{\partial^2 g}{\partial x^2} = 2x.$$

Then for

$$X_t = g(b_t)$$

we have

$$\begin{aligned} dX_t = d\left(\frac{b_t^3}{3}\right) &= \frac{1}{2}\frac{\partial^2 g}{\partial x^2}\, dt + \frac{\partial g}{\partial x}\, db_t \\ &= b_t\, dt + b_t^2\, db_t. \end{aligned}$$

Integrating from 0 to t we have

$$\begin{aligned} \int_0^t d\left(\frac{b_s^3}{3}\right) &= \int_0^t b_s ds + \int_0^t b_s^2\, db_s \\ \int_0^t b_s^2\, db_s &= \frac{b_t^3}{3} - \int_0^t b_s\, ds. \end{aligned}$$

* * *

Next we need to compute

$$\int_0^t s\, db_s.$$

Guessing

$$g(x) = tx$$

we have

$$\frac{\partial g}{\partial t} = x, \qquad \frac{\partial g}{\partial x} = t, \qquad \text{and} \qquad \frac{\partial^2 g}{\partial x^2} = 0.$$

Then for

$$X_t = g(t, b_t)$$

we have

$$\begin{aligned} dX_t = d(t\, b_t) &= \left(\frac{\partial g}{\partial t} + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}\right) dt + \frac{\partial g}{\partial x}\, db_t \\ &= b_t\, dt + t\, db_t. \end{aligned}$$

Integrating from 0 to t we have

$$\int_0^t d(s\, b_s) = \int_0^t b_s ds + \int_0^t s\, db_s$$
$$\int_0^t s\, db_s = t\, b_t - \int_0^t b_s\, ds.$$

* * *

The left-hand side in the original equation to be verified now becomes

$$3\int_0^t b_s^2\, db_s - 3\int_0^t s\, db_s = b_t^3 - 3\int_0^t b_s\, ds - 3t\, b_t + 3\int_0^t b_s\, ds$$
$$= b_t^3 - 3t\, b_t.$$

The right-hand side is

$$t^{3/2}\left(\frac{b_t}{t^{1/2}}\right)^3 - 3t^{3/2}\left(\frac{b_t}{t^{1/2}}\right) = b_t^3 - 3t\, b_t$$

as desired. $\square$

8 Exponential growth with multiple noise sources (problem 2.2)

Let $c, \alpha_1, \ldots, \alpha_n$ be constants; let $b_t = \left(b_t^{(1)}, \ldots, b_t^{(n)}\right)$ be n-dimensional Brownian motion. Let

$$X_t = \exp\left(ct + \sum_{j=1}^{n} \alpha_j b_t^{(j)}\right).$$

Prove that

$$dX_t = \left(c + \frac{1}{2}\sum_{j=1}^{n} \alpha_j^2\right) X_t\, dt + X_t \sum_{j=1}^{n} \alpha_j\, db_t^{(j)}.$$

Proof. Use the multidimensional Itō formula in the form of proposition 1.5. We have

$$\begin{aligned}
g(t, u_1, \ldots, u_n) &= e^{ct} e^{\alpha_1 u_1} \cdots e^{\alpha_n u_n} \\
\frac{\partial g}{\partial t}(t, u_1, \ldots, u_n) &= cg \\
\frac{\partial g}{\partial x_j}(t, u_1, \ldots, u_n) &= \alpha_j g \\
\frac{\partial g^2}{\partial x_j^2}(t, u_1, \ldots, u_n) &= \alpha_j^2 g.
\end{aligned}$$

Then

$$\begin{aligned}
dX_t &= \left(cX_t + \frac{1}{2}\sum_{j=1}^{n} \alpha_j^2 X_t\right) dt + \sum_{j=1}^{n} \alpha_j X_t\, db_t^{(j)} \\
&= \left(c + \frac{1}{2}\sum_{j=1}^{n} \alpha_j^2\right) X_t\, dt + X_t \sum_{j=1}^{n} \alpha_j\, db_t^{(j)}.
\end{aligned}$$

□

9 Martingales (problem 2.3)

9.1 Problem 2.3.a

Prove that

$$X_t = e^{t/2}\cos(b_t)$$

is a martingale with respect to the filtration of the Brownian motion b_t.

First recall that a stochastic process is a martingale with respect to a filtration $\mathcal{F}_s$ if

$$E[X_t \mid \mathcal{F}_s] = X_s.$$

This is an overblown, measure-theoretic way of obscuring the simple statement that, conditioned on $X_s = a$, the expected value of X_t for times past s is a.

For a concrete example, consider the d'Alembert system. The time t is in discrete steps $t = 0, 1, 2, \ldots$. One random variable, C_t, is the outcome of a coin flip with probability p of heads where we encode heads as $+1$ and tails as -1. The second random variable is

$$X_t = \sum_{s=0}^{t} C_s.$$

This represents the accumulated earnings of a gambler who, beginning with zero balance, plays a game wherein he/she wins a dollar for each head and loses a dollar for each tail. Now suppose that at the sth step the gambler has a balance of $a = 100$. (That is, condition on $X_s = 100$.) If the coin is fair, the expected value of X_{s+1} is $(101 + 99)/2 = 100$. Likewise $E[X_{s+2} = (102 + 2 \cdot 100 + 98)/4 = 100$ and so on. We say that for $t \geq s$, $E[X_t | X_s = a] = a$. One then generalizes this statement by considering not just one specific value of a but instead all values of X_s: we collect all the a's into a σ-algebra bag. We say

$$E[X_t \mid \mathcal{F}_s] = X_s.$$

Last, note that if the coin has a probability $p > 1/2$ of heads, then the expected future earnings, conditioned on the present balance, *exceed* the present balance. Similarly, if $p < 1/2$, then $E[X_t \mid F_s] < X_s$. The d'Alembert system is a martingale if and only if the coin is fair.

Brownian motion is simply (in a very specific sense) the limiting case of the d'Alembert system. When thinking of martingales with respect to the filtration of Brownian motion, I ask myself: for all processes X_t which pass through a at time s, what is the expected value of those processes at subsequent times t — is that expectation above, below, or at the current value a.

* * *

Returning to the original problem, recall two facts:

- If X is $\mathcal{F}$-measurable, then $E[X \mid \mathcal{F}] = X$.
- If X is independent of $\mathcal{F}$, then $E[X \mid \mathcal{F}] = E[X]$.

We split X_t into $(X_t - X_s) + X_s$. Then

$$E[X_t \mid \mathcal{F}_s] = E[X_t - X_s \mid \mathcal{F}_s] + E[X_s \mid \mathcal{F}_s].$$

Now $X_t - X_s$ is independent of $\mathcal{F}_s$ and X_s is $\mathcal{F}_s$-measurable. Thus

$$E[X_t \mid \mathcal{F}_s] = E[X_t] - E[X_s] + X_s.$$

We will be done if we can show $E[X_t] = E[X_s]$. Figure 7 shows forty realizations of X_t for t from 0 to 2 in steps of 0.002. The figure suggests that $E[X_t] = 1$ for all t. I have computed, but have not included in this paper, plots of $e^{ct}\cos(b_t)$ for values of c above and below 0.5: for $c < 0.5$, the mean $E[X_t]$ drops down from 1 as t increases from 0 (and X_t is a supermartingale); for $c > 0.5$, $E[X_t]$ rises from 1 (and X_t is a submartingale). Also note from the figure that, for each t, values of X_t occur with less spread, but more densely, above 1; they occur with more spread, but less densely, below 1. Thus the density function of X_t is asymmetrical but nonetheless has mean 1 for all t.

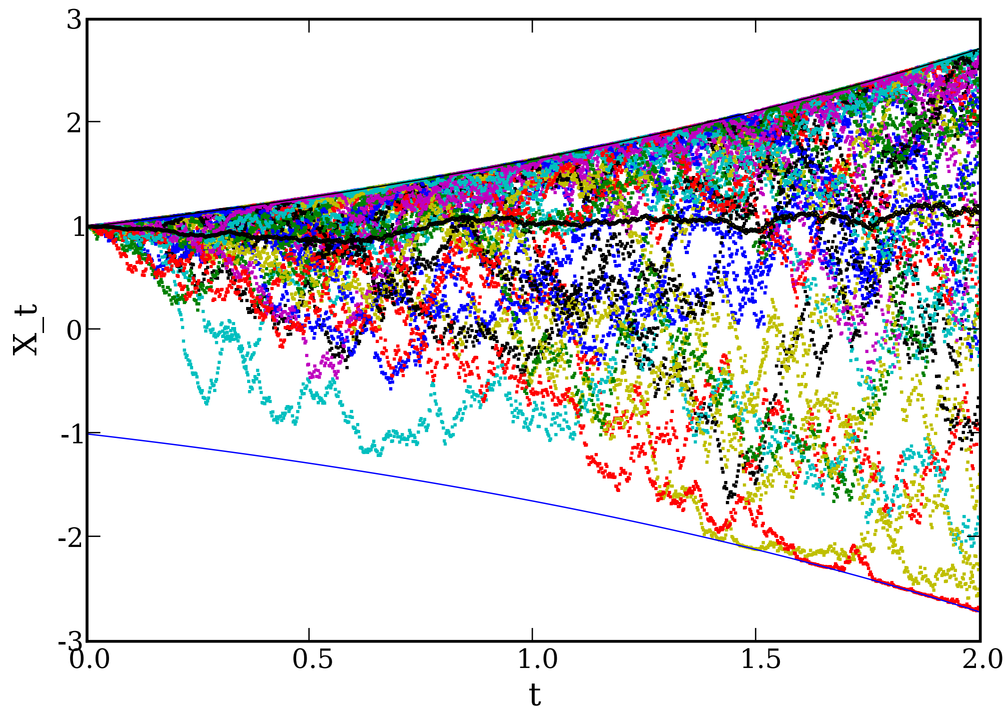


Figure 7: Problem 2.3.a: 40 realizations of $e^{t/2}\cos(b_t)$, with sample mean $\overline{X}_t$ and envelope curves $\pm e^{t/2}$.

Since

$$E[e^{t/2}\cos(b_t)] = e^{t/2}\, E[\cos(b_t)],$$

I need to show that $E[\cos(b_t)] = e^{-t/2}$. This is plausible intutively: at $t = 0$, b_t is 0 and $\cos(b_t) = 1$. As t increases slightly, the distribution of b_t spreads a little above and below 0, so $\cos(b_t)$ spreads a little below 1. As t becomes quite large and the distribution of b_t approaches uniformity on the real line, $\cos(b_t)$ approaches uniformity on $[-1, 1]$ and so the expectation of the latter approaches 0. This matches the behavior of $e^{-t/2}$ — as well as e^{-ct} in general; I need to show what's special about $c = 1/2$.

Since $E[X] = \int X\, dP$ for random variables X, and since the Law of the Unconscious Statistician tells me $E[f(X)] = \int f(X)\, dP$, and since b_t is normal with mean 0 and variance t, I can write down explicitly

$$E[\cos(b_t)] = \frac{1}{t\sqrt{2\pi}} \int_{-\infty}^{+\infty} \cos(x)\, e^{-\frac{x^2}{2t}}\, dx.$$

To solve this, observe that $\cos(b_t) = \mathrm{Re}(e^{ib_t})$. Employing the complete-the-square trick which is usual for Gaussian integrals, we have

$$\begin{aligned}
E[\cos(b_t)] &= \mathrm{Re}(E[e^{ib_t}]) \\
E[e^{ib_t}]) &= \frac{1}{t\sqrt{2\pi}} \int_{x=-\infty}^{x=+\infty} e^{ix} e^{-x^2/2t}\,dx \\
&= \frac{1}{t\sqrt{2\pi}} \int_{x=-\infty}^{x=+\infty} \exp\left[\frac{-1}{2t}\left(x^2 - 2tix - t^2 + t^2\right)\right] dx \\
&= \frac{e^{t^2/2t}}{t\sqrt{2\pi}} \int_{x=-\infty}^{x=+\infty} \exp\left[\frac{-1}{2t}(x - ti)^2\right] dx \\
&= \frac{e^{t/2}}{t\sqrt{2\pi}} \int_{x=-\infty}^{x=+\infty} \exp\left[\frac{-1}{2t}(x - ti)^2\right] dx.
\end{aligned}$$

I am done as long as

$$\mathrm{Re}\left(\int_{x=-\infty}^{x=+\infty} \exp\left[\frac{-1}{2t}(x - ti)^2\right] dx\right) = t\sqrt{2\pi}.$$

9.2 Problem 2.3.b

Prove that

$$X_t = (b_t + t)\exp(-b_t - t/2)$$

is a martingale with respect to the filtration of the Brownian motion b_t.

Just as above, we have

$$E[X_t \mid \mathcal{F}_s] = E[X_t] - E[X_s] + X_s;$$

again, it suffices to show that $E[X_t]$ is constant along t.

Figure 8 shows forty realizations of X_t. Many values of X_t drop far below the horizontal axis and have been clipped from the viewing window. Here X_t has much higher variance than in the previous problem, but it appears (from multiple computational runs) that the sample mean approaches 0 for all t.

As in the previous problem, I can combine the Law of the Unconscious Statistician with the known density function of b_t to compute $E[X_t]$:

$$\begin{aligned}
E[X_t] &= E[(b_t + t)e^{-b_t}e^{-t/2}] \\
&= e^{-t/2}E[(b_t + t)e^{-b_t}] \\
&= e^{-t/2}E[b_t\, e^{-b_t}] + e^{-t/2}E[t\, e^{-b_t}].
\end{aligned}$$

One computes

$$\begin{aligned}
E[b_t\, e^{-b_t}] &= \frac{1}{t\sqrt{2\pi}} \int_{-\infty}^{+\infty} x e^{-x} e^{-x^2/2t}\,dx \\
&= \frac{1}{t\sqrt{2\pi}} \left[-e^{t/2}\sqrt{\frac{\pi}{2}}\,\mathrm{erf}\left(\frac{t+x}{\sqrt{2t}}\right) t^{3/2} - te^{-x^2/2t-x}\right]_{x=-\infty}^{x=+\infty} \\
&= -\sqrt{t}e^{t/2}
\end{aligned}$$

since erf goes to ± 1 as $x \to \pm\infty$, respectively, and since $e^{-x^2/2t-x}$ goes to 0 as $x \to \pm\infty$.

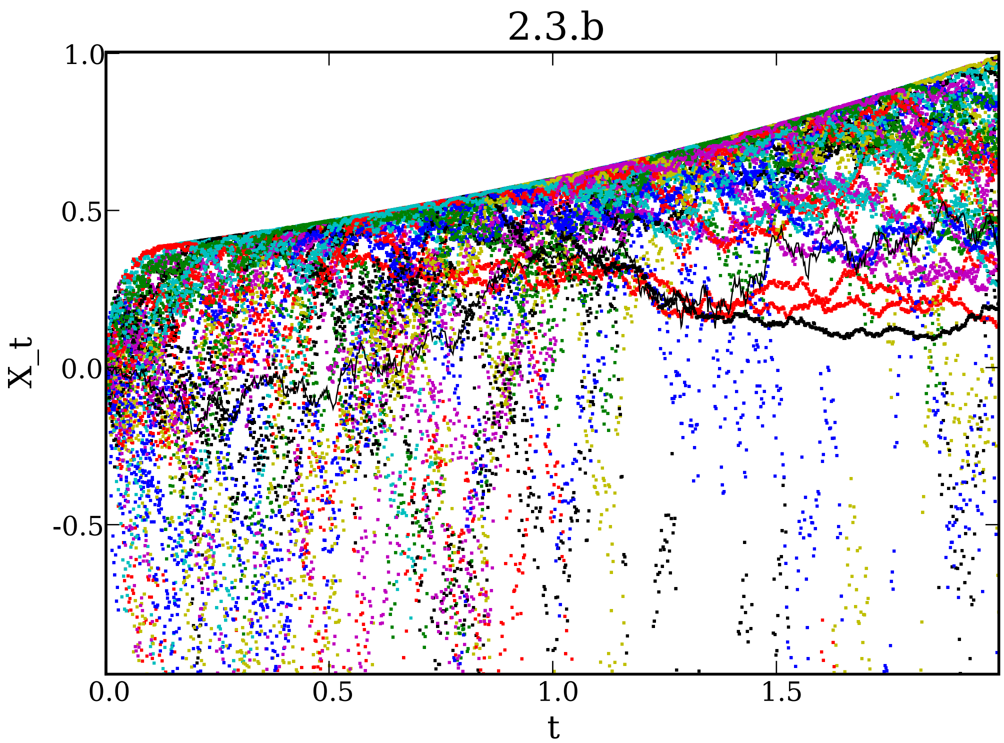


Figure 8: Problem 2.3.b: 40 realizations of $(b_t + t)\exp(-b_t - t/2)$ with sample mean $\overline{X}_t$.

Likewise, one computes

$$\begin{aligned} E[t\,e^{-b_t}] &= tE[e^{-b_t}] \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-x}e^{-x^2/2t}\,dx \\ &= \frac{1}{\sqrt{2\pi}}\left[e^{t/2}\sqrt{\frac{\pi}{2}}\,\mathrm{erf}\left(\frac{t+x}{\sqrt{2t}}\right)t^{1/2}\right]_{x=-\infty}^{x=+\infty} \\ &= \sqrt{t}e^{t/2}. \end{aligned}$$

Then

$$\begin{aligned} E[X_t] &= e^{-t/2}E[b_t\,e^{-b_t}] + e^{-t/2}E[t\,e^{-b_t}] \\ &= -e^{-t/2}\sqrt{t}e^{t/2} + e^{-t/2}\sqrt{t}e^{t/2} \\ &= 0 \end{aligned}$$

as desired.

10 Mean-reverting Ornstein-Uhlenbeck process (problem 2.4)

Solve the mean-reverting Ornstein-Uhlenbeck equation:

$$dX_t = (m - X_t)\,dt + \sigma\,db_t$$

with real constant m and positive real constant σ. Compute the mean and variance of the solution, using constant initial condition X_0.

Since this is first-order, an integrating-factor approach is again a good guess:

$$\begin{aligned} e^t\,dX_t &= e^t(m - X_t)\,dt + \sigma e^t\,db_t \\ d(e^t\,X_t) &= e^t X_t\,dt + e^t(m - X_t)dt + \sigma e^t\,db_t \\ &= me^t\,dt + \sigma e^t\,db_t. \end{aligned}$$

Integrating from 0 to t, we have

$$\begin{aligned} \int_0^t d(e^s\,X_s) &= m\int_0^t e^s\,ds + \sigma\int_0^t e^s\,db_s \\ e^t X_t - X_0 &= m(e^t - 1) + \sigma\int_0^t e^s\,db_s \\ X_t &= X_0 e^{-t} + m(1 - e^t) + \sigma\int_0^t e^{s-t}\,db_s. \end{aligned}$$

The mean is

$$\begin{aligned} E[X_t] &= X_0 e^{-t} + m(1 - e^{-t}) + \sigma E\left[\int_0^t e^{s-t}\,db_s\right] \\ &= X_0 e^{-t} + m(1 - e^{-t}) \end{aligned}$$

where the expectation of the integral vanishes by proposition 1.11.

For the variance, we first compute X_t^2:

$$\begin{aligned} X_t^2 &= X_0^2 e^{-2t} + m^2(1 - e^{-t})^2 + \sigma^2\left(\int_0^t e^{s-t}\,db_s\right)^2 \\ &+ 2mX_0 e^{-t}(1 - e^{-t}) + 2X_0 e^{-t}\sigma\int_0^t e^{s-t}\,db_s + 2m\sigma(1 - e^{-t})\int_0^t e^{s-t}\,db_s. \end{aligned}$$

The last two terms have zero expected value by proposition 1.11; the first, second, and fourth terms are non-stochastic and are their own means. For the third term I can again use the Itō isometry (proposition 1.10) with $f_s = e^{s-t}$:

$$\begin{aligned} E\left[e^{-2t}\left(\int_0^t e^s\,db_s\right)^2\right] &= E\left[e^{-2t}\int_0^t e^{2s}\,ds\right] = e^{-2t}\int_0^t e^{2s}\,ds \\ &= \frac{e^{-2t}}{2}\,\left[e^{2s}\right]_{s=0}^{s=t} = \frac{e^{-2t}}{2}\,[e^{2t} - 1] = \frac{1 - e^{-2t}}{2}. \end{aligned}$$

Combining terms, we have

$$E[X_t^2] = X_0^2 e^{-2t} + m^2(1 - e^{-t})^2 + 2mX_0(e^{-t} - e^{-2t}) + \frac{\sigma^2}{2}(1 - e^{-2t}).$$

Then

$$\begin{aligned}
\mathrm{Var}(X_t) &= E[X_t^2] - E[X_t]^2 \\
&= X_0^2 e^{-2t} + m^2(1-e^{-t})^2 + 2mX_0(e^{-t}-e^{-2t}) + \frac{\sigma^2}{2}(1-e^{-2t}) \\
&- X_0^2 e^{-2t} - m^2(1-e^{-t})^2 - 2mX_0(e^{-t}-e^{-2t}) \\
&= \frac{\sigma^2}{2}(1-e^{-2t}).
\end{aligned}$$

The interpretation of the mean and variance is that the process starts surely at X_0, then moves toward m: specifically, the initial state X_0 decays exponentially while the difference from the steady state m decays exponentially. Meanwhile the standard deviation of the process starts at zero, then increases to a steady-state value $\sigma/\sqrt{2}$.

Figure 9 shows six realizations of X_t with t from 0 to 10 in steps of 0.01. The parameters are $X_0 = 1.7$, $m = 2$, and $\sigma = 0.05$. In addition to the six realizations of the process, I also plot the mean $E[X_t]$, along with the mean plus or minus one and two standard deviations.

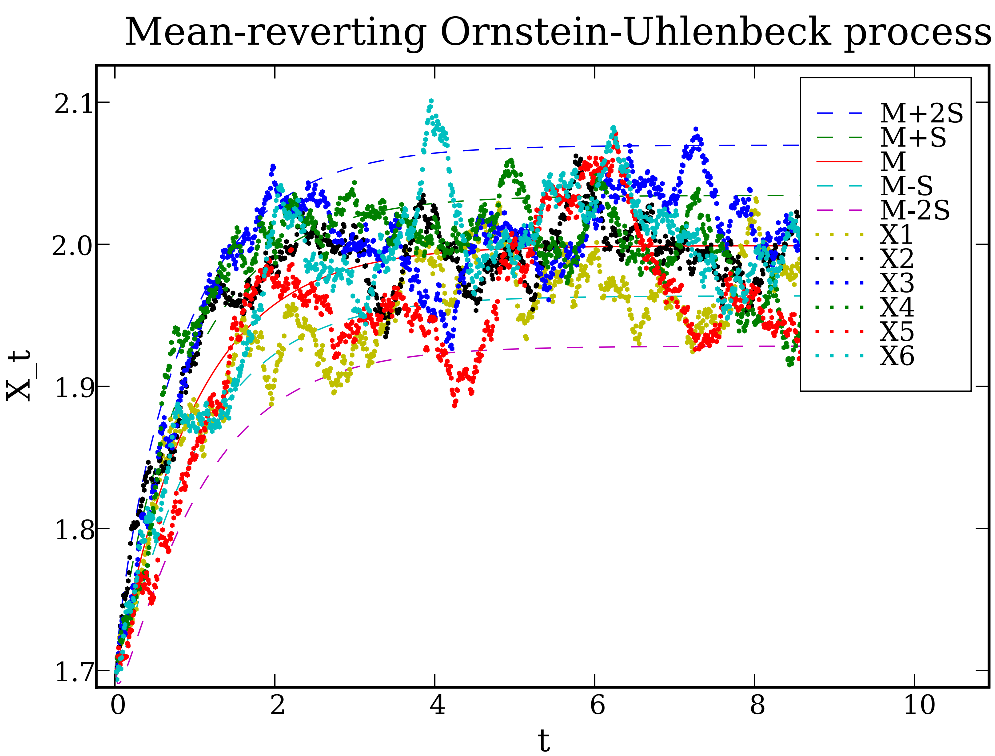


Figure 9: Problem 2.4: The mean-reverting Ornstein-Uhlenbeck process. $M = E[X_t]$; $S = \mathrm{Var}(X_t)^{1/2}$.

11 Stochastically forced vibrating string (problem 2.5)

Solve the stochastically forced vibrating-string equation:

$$dX_t = Y_t\,dt + \alpha\,da_t$$
$$dY_t = -X_t\,dt + \beta\,db_t$$

where α, β are real consants and a_t, b_t are independent Brownian motions.

On scratch paper, I tried using a pair of integrating factors — one for each equation — but this did not result in any simplification. Following a hint in [**Øks**], I realize that we may write

$$\begin{pmatrix} dX_t \\ dY_t \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} X_t\,dt \\ Y_t\,dt \end{pmatrix} + \begin{pmatrix} \alpha\,da_t \\ \beta\,db_t \end{pmatrix},$$

and that the appropriate integrating factor for this coupled (hence second-order) equation is e^{-At} where

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Writing down the power-series expansion

$$e^{-At} = \sum_{n=0}^{\infty} \frac{A^n}{n!}$$

and collecting even and odd terms, one recognizes the Maclaurin series for sine and cosine in the matrix elements, leading to

$$e^{-At} = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}.$$

Multiplying the coupled equations through by the integrating factor gives

$$\begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix} \begin{pmatrix} dX_t \\ dY_t \end{pmatrix} = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} X_t\,dt \\ Y_t\,dt \end{pmatrix} + \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix} \begin{pmatrix} \alpha\,da_t \\ \beta\,db_t \end{pmatrix}$$

$$\begin{pmatrix} \cos(t)\,dX_t - \sin(t)\,dY_t \\ \sin(t)\,dX_t + \cos(t)\,dY_t \end{pmatrix} = \begin{pmatrix} \sin(t) & \cos(t) \\ -\cos(t) & \sin(t) \end{pmatrix} \begin{pmatrix} X_t\,dt \\ Y_t\,dt \end{pmatrix} + \begin{pmatrix} \alpha\cos(t)\,da_t - \beta\sin(t)\,db_t \\ \alpha\sin(t)\,da_t + \beta\cos(t)\,db_t \end{pmatrix}$$

$$\begin{pmatrix} \cos(t)\,dX_t - \sin(t)\,dY_t \\ \sin(t)\,dX_t + \cos(t)\,dY_t \end{pmatrix} = \begin{pmatrix} \sin(t)X_t + \cos(t)Y_t \\ -\cos(t)X_t + \sin(t)Y_t \end{pmatrix} + \begin{pmatrix} \alpha\cos(t)\,da_t - \beta\sin(t)\,db_t \\ \alpha\sin(t)\,da_t + \beta\cos(t)\,db_t \end{pmatrix}. \tag{11.1}$$

Now, as in the one-dimensional problems above, we need to find

$$d\left(e^{-At}\begin{pmatrix} X_t \\ Y_t \end{pmatrix}\right) = d\begin{pmatrix} \cos(t)X_t - \sin(t)Y_t \\ \sin(t)X_t + \cos(t)Y_t \end{pmatrix}.$$

To use the Itō formula (proposition 1.4), write

$$g(t,x,y) = \cos(t)x - \sin(t)y$$
$$h(t,x,y) = \sin(t)x + \cos(t)y.$$

Then

$$\frac{\partial g}{\partial t} = -\sin(t)x - \cos(t)y, \qquad \frac{\partial g}{\partial x} = \cos(t), \qquad \frac{\partial g}{\partial y} = -\sin(t),$$
$$\frac{\partial h}{\partial t} = \cos(t)x - \sin(t)y, \qquad \frac{\partial h}{\partial x} = \sin(t), \qquad \frac{\partial h}{\partial y} = \cos(t),$$

and all second partials in x and/or y are zero. Then

$$\begin{array}{rcrcrcrcr} dg & = & -\sin(t)X_t\,dt & - & \cos(t)Y_t\,dt & + & \cos(t)\,dX_t & - & \sin_t\,dY_t \\ dh & = & \cos(t)X_t\,dt & - & \sin(t)Y_t\,dt & + & \sin(t)\,dX_t & + & \cos_t\,dY_t. \end{array} \tag{11.2}$$

Now we can re-write equation 11.1 as

$$\left(\begin{array}{c} \cos(t)\,dX_t - \sin(t)\,dY_t \\ \sin(t)\,dX_t + \cos(t)\,dY_t \end{array}\right) - \left(\begin{array}{c} \sin(t)\,X_t\,dt + \cos(t)\,Y_t\,dt \\ -\cos(t)\,X_t\,dt + \sin(t)\,Y_t\,dt \end{array}\right) = \left(\begin{array}{c} \alpha\cos(t)\,da_t - \beta\sin(t)\,db_t \\ \alpha\sin(t)\,da_t + \beta\cos(t)\,db_t \end{array}\right).$$

Using equation 11.2, we have

$$d\left(e^{-At}\begin{pmatrix} X_t \\ Y_t \end{pmatrix}\right) = \left(\begin{array}{c} \alpha\cos(t)\,da_t - \beta\sin(t)\,db_t \\ \alpha\sin(t)\,da_t + \beta\cos(t)\,db_t \end{array}\right).$$

Integrating both sides with respect to t, we obtain

$$\int_0^t d\left(e^{-As}\begin{pmatrix} X_s \\ Y_s \end{pmatrix}\right) = \left(\begin{array}{c} \alpha\int_0^t \cos(s)\,da_s - \beta\int_0^t \sin(s)\,db_s \\ \alpha\int_0^t \sin(s)\,da_s + \beta\int_0^t \cos(s)\,db_s \end{array}\right).$$

$$e^{-At}\left(\begin{array}{c} X_t \\ Y_t \end{array}\right) = \left(\begin{array}{c} X_0 \\ Y_0 \end{array}\right) + \left(\begin{array}{c} \alpha\int_0^t \cos(s)\,da_s - \beta\int_0^t \sin(s)\,db_s \\ \alpha\int_0^t \sin(s)\,da_s + \beta\int_0^t \cos(s)\,db_s \end{array}\right).$$

$$\left(\begin{array}{c} X_t \\ Y_t \end{array}\right) = \left(\begin{array}{cc} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{array}\right)\left(\begin{array}{c} X_0 + \alpha\int_0^t \cos(s)\,da_s - \beta\int_0^t \sin(s)\,db_s \\ Y_0 + \alpha\int_0^t \sin(s)\,da_s + \beta\int_0^t \cos(s)\,db_s \end{array}\right).$$

Figure 10 shows a realization of X_t and Y_t for t from 0 to 100 in steps of 0.01 and $\alpha = \beta = 0.01$: the upper plot shows shows X_t Y_t as functions of time, while the lower plot is a phase portrait. Figure 11 is similar, but with $\alpha = \beta = 0.1$.

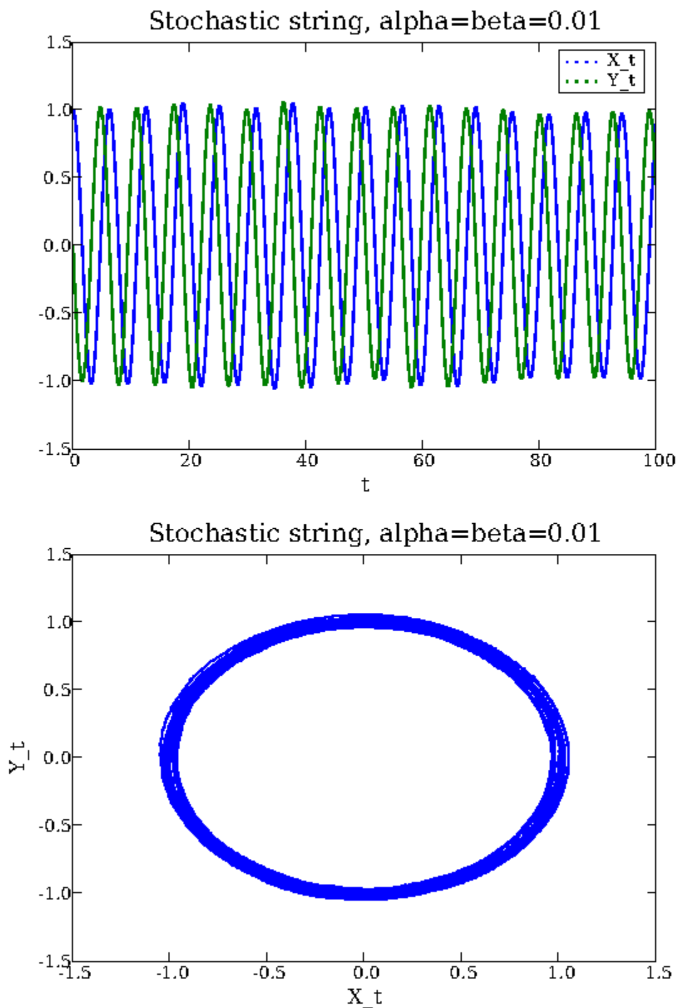
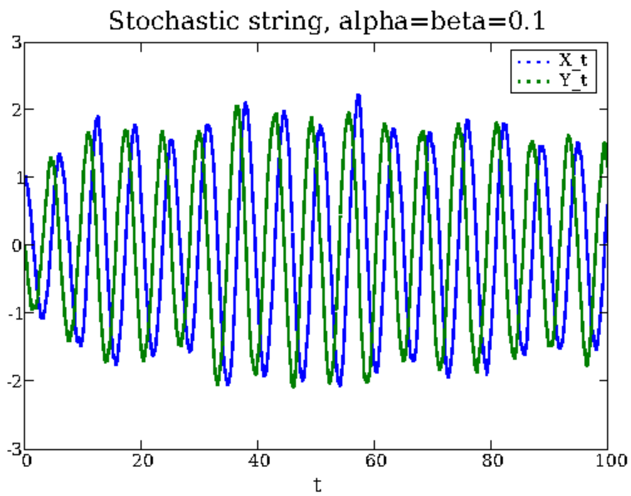


Figure 10: Problem 2.5: Weakly stochastically forced vibrating string.



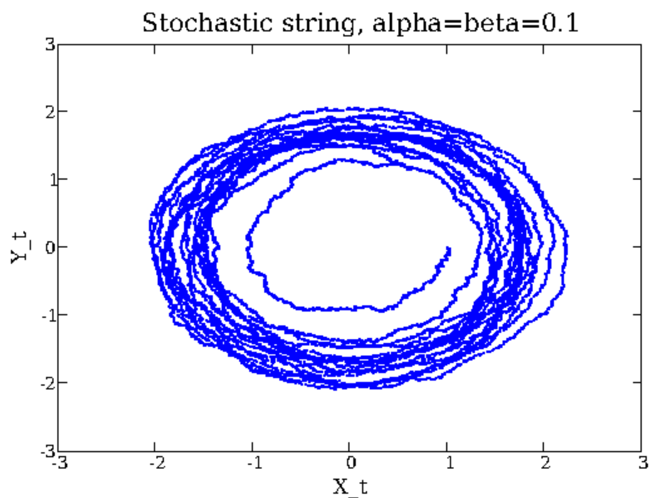


Figure 11: Problem 2.5: Strongly stochastically forced vibrating string.

12 The Feynman-Kac formula

The following was not contained in Dr. Wehr's problem set, but (as I write this section in the summer of 2008) is useful background material for my comprehensive exam. The proof is adapted from [**Law**], with missing steps filled in and notation changed to match that used in this paper. (See also [**Øks**] where uniqueness of the solution is proved.)

Proposition 12.1. *The solution to the partial differential equation*

$$\frac{\partial \phi}{\partial t}(t,x) = \frac{f(x)^2}{2}\frac{\partial^2 \phi}{\partial x^2}(t,x) + e(x)\frac{\partial \phi}{\partial x}(t,x) + g(x)\phi(t,x) \tag{12.1}$$

with initial condition

$$\phi(0,x) = h(x)$$

is

$$\phi(t,x) = \mathbb{E}^x\left[h(X_t)\exp\left\{\int_0^t g(X_u)\,du\right\}\right], \tag{12.2}$$

where X_t satisfies

$$dX_t = e(X_t)\,dt + f(X_t)\,db_t$$

and $\mathbb{E}^x[Y]$ denotes $\mathbb{E}[Y \mid X_0 = x]$. (We assume that this expectation exists for all t, x.)

Remark 12.2. We call X_t a *time homogeneous diffusion*, or simply a *diffusion*, since e and f depend on X_t but not on t directly. Note that X_t is Markovian.

Proof. Let $\phi(t,x)$ be given by equation 12.2. We will show that it satisfies 12.1. For brevity, write

$$J_t = \exp\left\{\int_0^t g(X_u)\,du\right\}.$$

Note that for $s < t$,

$$J_t = \exp\left\{\int_0^t g(X_u)\,du\right\} = \exp\left\{\int_0^s g(X_u)\,du\right\}\exp\left\{\int_s^t g(X_u)\,du\right\} = J_s\exp\left\{\int_s^t g(X_u)\,du\right\}.$$

If $s < t$ then, since J_s is $\mathcal{F}$-measurable,

$$\begin{aligned}\mathbb{E}[h(X_t)J_t \mid \mathcal{F}_s] &= J_s\mathbb{E}\left[h(X_t)\exp\left\{\int_s^t g(X_u)\,du\right\} \mid \mathcal{F}_s\right] \\ &= J_s\phi(t-s, X_s).\end{aligned}$$

Call this quantity M_s. We claim that it is a martingale for $0 \le s < t$. To see this, suppose $r < s$. Then

$$\mathbb{E}\left[\mathbb{E}\left[h(X_t)J_t \mid \mathcal{F}_s\right] \mid \mathcal{F}_r\right] = \mathbb{E}[h(X_t)J_t \mid \mathcal{F}_r].$$

Using the stochastic product rule (proposition 1.7), we have

$$dM_s = J_s\,d\phi(t-s, X_s) + dJ_s\,\phi(t-s, X_s) + dJ_s\,d\phi(t-s, X_s).$$

For the first differential, using the chain rule and the second fundamental theorem of calculus,

$$dJ_s = \frac{\partial}{\partial s}\left(\exp\left\{\int_0^s g(X_u)\,du\right\}\right)\,ds = \exp\left\{\int_0^s g(X_u)\,du\right\} g(X_s)\,ds = J_s\,g(X_s)\,ds.$$

For the second differential, applying the Itō formula (assuming sufficient differentiability) in the form of equation 1.3 to $Y_s = \phi(t-s, X_s)$, we obtain

$$\begin{aligned} d\phi(t-s, X_s) = &\left(\frac{\partial \phi}{\partial s}(t-s, X_s) + e(X_s)\,\frac{\partial \phi}{\partial x}(t-s, X_s) + \frac{f(X_s)^2}{2}\frac{\partial^2 \phi}{\partial x^2}(t-s, X_s)\right)\,ds \\ &+ f(X_s)\,\frac{\partial \phi}{\partial x}(t-s, X_s)\,db_s. \end{aligned}$$

The third product term $dJ_s\,d\phi(t-s, X_s)$ is zero since its differentials are of the form ds^2 and $ds\,db_s$. Then

$$\begin{aligned} dM_s = &J_s\,\left(\frac{\partial \phi}{\partial s}(t-s, X_s) + e(X_s)\,\frac{\partial \phi}{\partial x}(t-s, X_s) + \frac{f(X_s)^2}{2}\frac{\partial^2 \phi}{\partial x^2}(t-s, X_s)\right)\,ds \\ &+ J_s\,f(X_s)\,\frac{\partial \phi}{\partial x}(t-s, X_s)\,db_s + J_s\,g(X_s)\,\phi(t-s, X_s)\,ds. \end{aligned}$$

Since M_s is a martingale, the ds terms must sum to zero. There is a common factor of J_s, but the exponential function is everywhere non-zero so the following four terms must be zero:

$$\begin{aligned} 0 &= \frac{\partial \phi}{\partial s}(t-s, X_s) + e(X_s)\frac{\partial \phi}{\partial x}(t-s, X_s) + \frac{f(X_s)^2}{2}\frac{\partial^2 \phi}{\partial x^2}(t-s, X_s) + g(X_s)\phi(t-s, X_s) \\ -\frac{\partial \phi}{\partial s}(t-s, X_s) &= \frac{f(X_s)^2}{2}\frac{\partial^2 \phi}{\partial x^2}(t-s, X_s) + e(X_s)\frac{\partial \phi}{\partial s}(t-s, X_s) + g(X_s)\phi(t-s, X_s). \end{aligned}$$

Putting $s = 0$ and recalling $X_0 = x$, we are done:

$$\frac{\partial \phi}{\partial t}(t, x) = \frac{f(x)^2}{2}\frac{\partial^2 \phi}{\partial x^2}(t, x) + e(x)\frac{\partial \phi}{\partial x}(t, x) + g(x)\phi(t, x).$$

□

Figure 12 plots equation 12.2 for x from -2 to 2 in steps of 0.1, for $t = 0, 1, 2$, $f = \sqrt{2}$, $e = 1$, $g = 0$, and

$$h(x) = \frac{1}{1 + x^2}.$$

That is, we are estimating the solution to

$$\frac{\partial \phi}{\partial t}(t, x) = \frac{\partial^2 \phi}{\partial x^2}(t, x) + \frac{\partial \phi}{\partial x}(t, x).$$

The expectations are taken over 10,000 Brownian motions for each x; these are stepped with $\Delta t = 0.01$. Since the coefficient on $\partial^2\phi/\partial x^2$ is nonzero, we expect the initial condition to spread over time; since the coefficient on $\partial\phi/\partial x$ is nonzero, we expect drift.

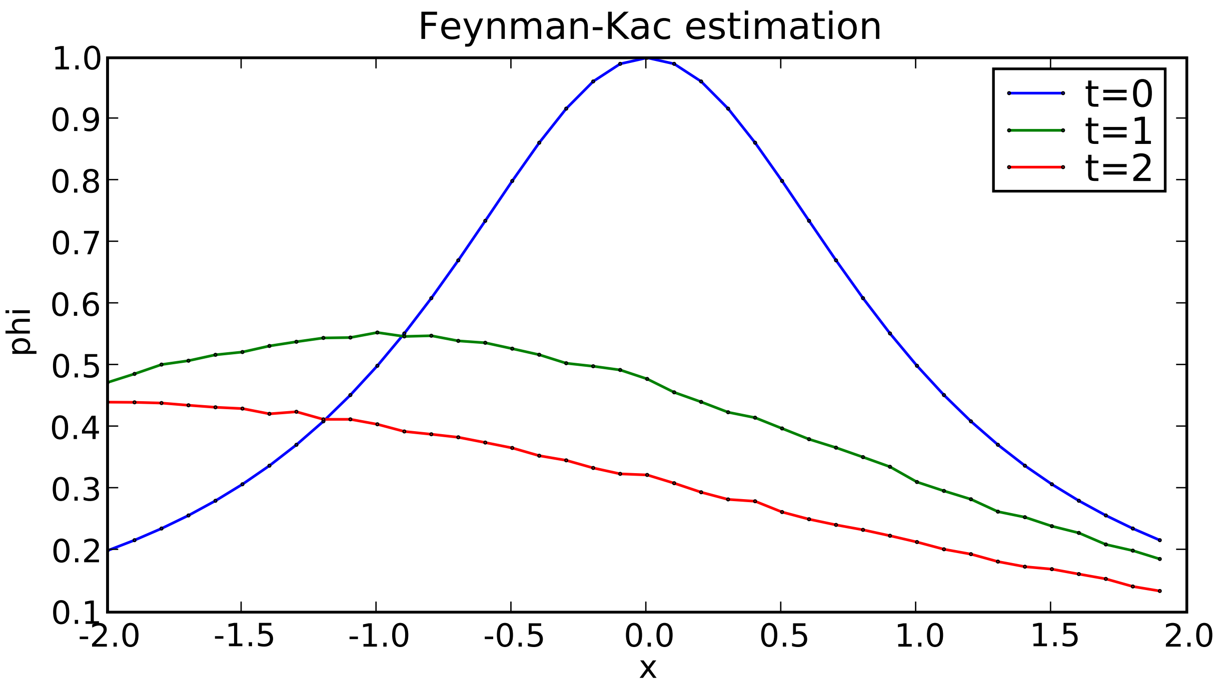


Figure 12: Feynman-Kac solution for diffusion with drift.

13 Generators and forward equations

As with section 12, the following was not contained in Dr. Wehr's problem set, but (as I write this section in the summer of 2008) is useful background material for my comprehensive exam.

The plan of this section is as follows:

(i) Present an Ornstein-Uhlenback process which is will be useful for comparing the generator and the forward equation.

(ii) Define the generator, and compute it for the O-U process.

(iii) Define the forward equation, and compute it for the O-U process.

(iv) Compare and contrast.

13.1 Scaled Brownian motion, and another Ornstein-Uhlenbeck process

Definition 13.1. *Scaled Brownian motion* is

$$Y_t = \sigma b_t.$$

Definition 13.2. The *decay-controlled Ornstein-Uhlenbeck process* satisfies the SDE

$$dZ_t = -\beta\, Z_t\, dt + \sigma\, db_t.$$

Compare this DCOU process with the mean-reverting Ornstein-Uhlenbeck (MROU) process in section 10. I will not include plots here. The DCOU decays to mean 0; the parameter β allows one to control the quickness of that decay.

To solve it, we try the integrating factor $e^{\beta t}$:

$$\begin{aligned} e^{\beta t}\, dZ_t &= -\beta e^{\beta t}\, Z_t\, dt + \sigma e^{\beta t}\, db_t \\ d(e^{\beta t} Z_t) &= \beta e^{\beta t} Z_t\, dt - \beta e^{\beta t} Z_t\, dt + \sigma e^{\beta t}\, db_t = \sigma e^{\beta t}\, db_t \\ \int_0^t e^{\beta s} Z_s\, ds &= \sigma \int_0^t e^{\beta s}\, db_s \\ Z_t &= Z_0 e^{-\beta t} + \sigma \int_0^t e^{-\beta(t-s)}\, db_s. \end{aligned}$$

Doing computations similar to those in section 10, we find that Z_t has mean and variance

$$\begin{aligned} \mathbb{E}[Z_t] &= Z_0 e^{-\beta t} \\ \mathrm{Var}(Z_t) &= \frac{\sigma^2}{2\beta}(1 - e^{-2\beta t}). \end{aligned}$$

Definition 13.3. For a stochastic process

$$dX_t = e(X_t, t)\, dt + f(X_t, t)\, db_t,$$

the **instantaneous mean** is $e(X_t, t)$ and the **instantaneous variance** is $f(X_t, t)$.

Thus, scaled Brownian motion has instantaneous mean 0 and instantaneous variance σ^2, but mean 0 and variance $\sigma^2 t$. The DCOU process has instantaneous mean $-\beta Z_t$ and instantaneous variance σ^2, but mean and variance as shown above. These computations make it clear that the instantaneous mean and instantaneous variance of a process are distinct from its mean and variance.

13.2 Generator of a diffusion process

Definition 13.4. Let the stochastic process X_t be defined by the SDE

$$dX_t = e(X_t, t)\, dt + f(X_t, t)\, db_t.$$

As in remark 12.2, we say that X_t is a **diffusion process** if e and f are time-independent, i.e. if we may write

$$dX_t = e(X_t)\, dt + f(X_t)\, db_t.$$

Definition 13.5. Let X_t be a diffusion process as above. Let $\phi(X_t)$ be some function of the process. (In mathematical physics, ϕ might be some observable.) Define

$$f_t(x) = \mathbb{E}^x[\phi(X_t)],$$

where

$$\mathbb{E}^x[\cdot] = \mathbb{E}[\cdot \mid X_0 = x].$$

Then $f_t(x)$ satisfies

$$\frac{\partial f_t}{\partial t} = \mathcal{L} f_t$$

where $\mathcal{L}$ is a differential operator called the **generator** of the process X_t.

Here we compute the generators of scaled Brownian motion and the DCOU process. As shown in [**Øks**], the generator $\mathcal{L}$ of a diffusion process

$$dX_t = e(X_t)\,dt + f(X_t)\,db_t$$

is

$$\mathcal{L} = e(x)\frac{\partial}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}.$$

For scaled Brownian motion, which satisfies $dX_t = db_t$ so $e = 0$ and $f = \sigma$, we have

$$\mathcal{L}_{\mathrm{B}} = \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}. \tag{13.1}$$

(In $d \geq 1$ dimensions, this becomes $\frac{1}{2}\nabla^2$.) For the DCOU process, with $e(x) = -\beta x$ and $f(x) = \sigma$, we have

$$\mathcal{L}_{\mathrm{DCOU}} = -\beta x\,\frac{\partial}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}. \tag{13.2}$$

13.3 The forward equation

Proposition 13.6. *Let X_t be a stochastic process (not necessarily a diffusion) satisfying the SDE*

$$dX_t = e(X_t, t)\,dt + f(X_t, t)\,db_t,$$

with initial condition distributed according to

$$X_0(x) \sim g(x, 0).$$

*Then the density $g(x,t)$ of X_t satisfies the **forward equation** or **Fokker-Planck equation** of X_t which solves*

$$\frac{\partial g(x,t)}{\partial t} = -\frac{\partial}{\partial x}\left[e(x,t)g(x,t)\right] + \frac{\partial^2}{\partial x^2}\left[\frac{f(x,t)^2}{2}g(x,t)\right].$$

Remark. The forward equation governs the time evolution of the density: it is the PDE of the PDF.

Remark. The term $e(x,t)$ is called the **drift** of the process; the term $\frac{1}{2}f(x,t)^2$ is called the **diffusion**. Note that the drift is synonymous with the instantaneous mean; the diffusion is half the instantaneous variance.

Proof. See [**Øks**]. □

For scaled Brownian motion, with $e(x,t) = 0$ and $f(x,t) = \sigma$, we have the forward equation

$$\frac{\partial g(x,t)}{\partial t} = \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}\left[g(x,t)\right].$$

For the DCOU process, with $e(x,t) = -\beta x$ and $f(x,t) = \sigma$, we have

$$\frac{\partial g(x,t)}{\partial t} = \beta\frac{\partial}{\partial x}\left[xg(x,t)\right] + \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}\left[g(x,t)\right].$$

13.4 Comparison

The generator and forward equation are superficially similar, but distinct. In particular, while they coincide for the Brownian motion, they differ for the DCOU process.

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